MT804 Analysis Homework II

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p. 135 4.5.1, 4.5.2 p. 136 4.5.3 (part a only) p. 140 4.6.1

Exercise 4.5.1 Use the Intermediate Value Theorem to prove that every polynomial of with real coefficients and odd degree has a zero in $\mathbb{R}$.

Proof: Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with n odd and all $a_i \in \mathbb{R}$, with $a_n \neq 0$. Dividing $p(x)$ by a_n, we may assume $a_n = 1$. Let $q(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, so that

$$p(x) = x^n \left(1 + \frac{q(x)}{x^n}\right).$$

Since $q(x)$ has degree $< n$, the term in parentheses goes to 1 as $|x| \to \infty$. Choose $b > 0$ such that $1 + \frac{q(x)}{x^n} > 0$ for $|x| \geq b$. Then

$$p(b) = b^n \left(1 + \frac{q(b)}{b^n}\right) > 0,$$

while

$$p(-b) = -b^n \left(1 + \frac{q(-b)}{(-b)^n}\right) < 0,$$

since n is odd. By the I.V.T., there is a zero of $p(x)$ in the interval $[-b, b]$. ■

Exercise 4.5.2 Let $I_0 = [a_0, b_0]$, $a_0 < b_0$, and let $f \in C(I)$ be continuous with $f(a_0) < 0 < f(b_0)$. Use the bisection method to prove that there exists $z \in [a_0, b_0]$ with $f(z) = 0$. Then use this method find n such that the midpoint z_n approximates $\sqrt{2}$ to ten decimal places.

Proof: Let z_0 be the midpoint of $[a_0, b_0]$. If $f(z_0) = 0$, we're done. Otherwise, define a new interval $[a_1, b_1]$ by:

$$[a_1, b_1] = \begin{cases} [a_0, z_0] & \text{if } f(z_0) > 0 \\ [z_0, b_0] & \text{if } f(z_0) < 0 \end{cases}$$

so that we again have $f < 0$ at the left endpoint and $f > 0$ at the right endpoint. Let z_1 be the midpoint of $[a_1, b_1]$. Repeating this process, we may eventually get a zero of f at one of the midpoints z_n, in which case we have our zero and the process stops. Assume that $f(z_n) \neq 0$ for all n.

We therefore have a nested sequence of closed intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots$$

each half the length of the previous one. So

$$b_n - a_n = (b_0 - a_0) \cdot 2^{-n} \to 0.$$

By the Nested Intervals Theorem, there is a number

$$z \in \bigcap_{n=0}^{\infty} [a_n, b_n].$$

If z' is any point in this intersection and $\epsilon > 0$ then choose n that $b_n - a_n < \epsilon$. Since z, z' are both in $[a_n, b_n]$ we have $|z - z'| < \epsilon$. Since ϵ was arbitrary, this means $z' = z$. We have shown that

$$\bigcap_{n=0}^{\infty} [a_n, b_n] = \{z\}.$$

It remains to prove that $f(z) = 0$. Suppose $f(z) \neq 0$. Then the number $\epsilon = |f(z)|$ is positive.

Since f is uniformly continuous on I, there exists $\delta > 0$ such that for all $x, x' \in I$ with $|x - x'| < \delta$ we have $|f(x) - f(x')| < \epsilon = |f(z)|$. Choose n large enough that $b_n - a_n < \delta$.

If $f(z) > 0$, take $x = z, x' = a_n$. Since $z \in [a_n, b_n]$, we have $|z - a_n| < \delta$ so

$$f(z) - f(a_n) = |f(z) - f(a_n)| < \epsilon = f(z),$$

so $f(a_n) > 0$, a contradiction.

If $f(z) < 0$, take $z = b_n$, $x' = z$. Again we have $|b_n - z| < \delta$, so

$$f(b_n) - f(z) = |f(b_n) - f(z)| < \epsilon = -f(z),$$

so $f(b_n) < 0$, another contradiction. It follows that $f(z) = 0$. ■

Now let's approximate $\sqrt{2}$. Take $f(x) = x^2 - 2$ on the interval $[0, 2]$. Since each of the midpoints will be averages of rational numbers, hence rational, and $\sqrt{2}$ is irrational, we get an infinite sequence of intervals as above. We want

$$(b_0 - a_0) \cdot 2^{-n} < 10^{-10},$$

or

$$2^n > 10^{10},$$

or

$$n > 10 \cdot \frac{\log 10}{\log 2} = 33.2 \cdots .$$

So the rational number z_{34} approximates $\sqrt{2}$ to ten decimal places.

Exercise 4.5.3a Show that any two non-empty open intervals are homeomorphic.

Proof: Since homeomorphisms are invertible, it suffices to show that any nonempty open interval I is homeomorphic to $(-1, 1)$. First suppose $I = (a, b)$ is bounded. We construct an affine function $f(x) = \alpha x + \beta$ so that $f(a) = 0$ and $f(b) = 1$. This gives two equations to be solved for α and β and we get the function

$$f(x) = \frac{2x - a - b}{b - a}$$

which is a homeomorphism $f : (a, b) \to (-1, 1)$. Indeed, the inverse of f is

$$f^{-1}(x) = \tfrac{1}{2}[(b - a)x + a + b].$$

Now suppose I is unbounded. If $I = \mathbb{R}$ then the function $f(x) = x/\sqrt{1 + x^2}$ is a homeomorphism $f : \mathbb{R} \to (-1, 1)$, as shown in the text, p. 136. The remaining two cases are $I = (-\infty, b)$ and $I = (a, \infty)$. Each of the first kind $I = (-\infty, b)$ is homeomorphic to $(-b, \infty)$ via the map $x \mapsto -x$ and $(-b, \infty)$ is of the second kind. So it suffices to find a homeomorphism from (a, ∞) to $(-1, 1)$. Here's a nice one:

$$f(x) = \frac{x - a - 1}{x - a + 1}, \qquad f^{-1}(x) = \frac{(1 - a)x + a + 1}{1 - x}.$$

Note that the discontinuity of f is at $a - 1$, which is not in (a, ∞). Likewise, the discontinuity of $f^{-1}(x)$ is at $x = 1$ which is not in $(-1, 1)$.

Exercise 4.6.1 Prove that $\sin(1/x)$ is not uniformly continuous on $(0, 1]$.

Proof: By the Continuous Extension Theorem (Thm 4.6.2) it suffices to show that $\sin(1/x)$ does not extend to a continuous function on $[0, 1]$. Let $c \in \mathbb{R}$ and suppose we define $f(x) = \sin(1/x)$ on $(0, 1]$ and $f(0) = c$. Let $x_n = 1/n\pi$ and let $x'_n = 1/(2n\pi + \pi/2)$. Then $f(x_n) = 0$ and $f(x'_n) = 1$ for all n. So, no matter what c is, we cannot have $\lim_{x\to 0} f(x) = c$. Hence $f(x)$ is discontinuous at $x = 0$ and there is no continuous extension of $\sin(1/x)$ to $(0, 1]$. ■

Extra Exercise 1 Let K be a compact subset of $\mathbb{R}$. Prove that $\sup(K)$ and $\inf(K)$ exist and belong to K.

Proof: By the Heine-Borel theorem, K is both closed and bounded. Since K is bounded, the real numbers $s = \sup(K)$ and $\ell = \inf(K)$ exist. Suppose s is not in K. Then $s \in K^c$, which is open, since K is closed. Choose $\epsilon > 0$ such that $V_{2\epsilon}(s) \subset K^c$. Then $s - \epsilon$ is an upper bound for K, contradicting the fact that s is the least upper bound of K. A similar argument shows that $\ell \in K$, as well. ■

Extra Exercise 2 Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ is rational in lowest terms and } n > 0 \\ 0 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Prove that f is continuous at every irrational number and discontinuous at every rational number.

Proof: First, if $x = m/n \in \mathbb{Q}$ then $f(x) = 1/n > 0$. Choose a sequence (x_k) in $\mathbb{Q}^c$ converging to x. Then $f(x_k) = 0$ for all k, so $\lim_k f(x_k) = 0 \neq f(x)$. The case where $x \in \mathbb{Q}^c$ is trickier. Since x is not an integer, there is $N \in \mathbb{N}$ such that $x \in (N, N + 1)$. Let $\epsilon > 0$. Since $f(x) = 0$, we must find $\delta > 0$ such that $|f(x')| < \epsilon$ for all $x' \in V_\delta(x)$.

I claim that there are only finitely many points in $(N, N + 1)$ where this can fail. Indeed, suppose $y \in (N, N + 1)$ and $f(y) \geq \epsilon$. Since $f(y) > 0$, we must have $y \in \mathbb{Q}$, say $y = m/n$ in lowest terms with $n > 0$, and $f(y) = 1/n \geq \epsilon$. This means $n \leq 1/\epsilon$, so there are only finitely many possibilities for the integer

n. Also, since $y \in (N, N+1)$ we have

$$N < \frac{m}{n} < N+1,$$

so

$$nN < m < n(N+1).$$

with N fixed and only finitely many possibilities for n, we see there are only finitely many possibilities for m as well. Hence there are only finitely many $y \in (N, N+1)$ for which $f(y) \geq \epsilon$. Since $f(x) = 0$, the number δ defined by

$$\delta := \min\{|x-N|, |x-N-1|, |x-y| : \ y \in (N, N+1), \text{ and } f(y) \geq \epsilon\}$$

is positive and for $|x - x'| < \delta$ we have $f(x') < \epsilon$. ■