MT804 Analysis Homework III

Solutions

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Exercise 6.4.2a Use the Mean Value Theorem and the fact that $(e^x)' = e^x$ to prove that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$

Proof: Let $f(x) = e^x - x - 1$. We have $f(0) = 0$, and $f'(x) = e^x - 1$ has the same sign as x. Let $0 \neq x \in \mathbb{R}$. By the MVT there exists c between 0 and x such that $f(x) = f'(c)x$. Since $f'(c)$ has the same sign as c which has the same sign as x, we have $f'(c)x > 0$. Hence $f(x) > 0$. ■

Exercise 6.4.2c Use the Mean Value Theorem and the power rule to prove that $(1 + x)^r > 1 + rx$ for all $x > -1$ and $r > 1$.

Proof: Let $f(x) = (1 + x)^r - rx - 1$. For $0 \neq x \in (-1, \infty)$, choose c between 0 and x such that $f(x) - f(0) = f'(c)(x - 0)$. Since $f(0) = 0$ and $f'(x) = r[(1 + x)^{r-1} - 1]$ has the opposite sign as x, it follows that $f(x) = f'(c)x > 0$. ■

Exercise 6.4.4 (Gronwall) Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. If $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in (0, \infty)$, show that $f(x) = 0$ for all $x \geq 0$.

Proof: Following the hint, we differentiate the function $g(x) = f(x)^2 e^{-2x}$ and find that

$$g'(x) = 2f^2 e^{-2x} \left(\frac{f'}{f} - 1 \right).$$

Since $|f| \leq |f|$ we have $|g'| \leq 0$ on $[0, \infty)$. Hence

$$g(x) = g(x) - g(0) = g'(c_x)x \leq 0.$$

But $g(x) \geq 0$ for all x, since f^2 and e^{-2x} are ≥ 0. It follows that $g(x) = 0$ for all $x \geq 0$, so $f(x) = 0$ for all $x \geq 0$. ■

Exercise 6.4.5 Consider $f(x) = x - x^2$ and $g(x) = 2x^3 - 3x^4$ on $[0, 1]$. Show there is no $c \in (0, 1)$ for which

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)}.$$

Answer: The equation to be solved is

$$\frac{0}{-1} = \frac{1}{6c^2},$$

which has no solution. However, Cauchy's MVT asserts that there is $c \in (0, 1)$ such that

$$g'(c)[f(1) - f(0)] = f'(c)[g(1) - g(0)],$$

which works out to

$$(6c^2 - 12c^3) \cdot 0 = (1 - 2c) \cdot (-1),$$

which has the solution $c = 1/2$.

Exercise 6.5.2 Find each limit

$$a)\ \lim_{x \to 0+} \frac{e^{-1/x}}{x}, \qquad b)\ \lim_{x \to 0+} (\sin x)^x.$$

Answer:

$$\lim_{x \to 0+} \frac{e^{-1/x}}{x} = \lim_{t \to \infty} \frac{e^{-t}}{1/t} = \lim_{t \to \infty} \frac{t}{e^t} = \lim_{t \to \infty} \frac{1}{e^t} = 0.$$

$$\lim_{x \to 0+} x \log \sin x = \lim_{x \to 0+} \frac{\log \sin x}{1/x} = \lim_{x \to 0+} \frac{-x^2 \cos x}{\sin x} = \lim_{x \to 0+} \frac{-x}{\sin x} \cdot x \cos x = -1 \cdot 0 = 0,$$

so

$$\lim_{x \to 0+} (\sin x)^x = \lim_{x \to 0+} e^{x \log \sin x} = 1.$$

Exercise 6.6.3b Show that the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

belongs to $C^\infty(\mathbb{R})$ and satisfies $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

Proof: It suffices to show that $f^{(n)}(0)$ exists and equals zero for all n. The same method as 6.5.2 a) shows that

$$\lim_{x \to 0+} \frac{e^{-1/x}}{x^n} = 0$$

for any $n \in \mathbb{N}$. By induction, we find that

$$f^{(n)}(x) = \frac{p_n(x)}{x^{2n}} e^{-1/x}, \qquad \text{for} \quad x > 0,$$

where $p_n(x)$ is a polynomial. We have $f(0) = 0$. Assume $f^{(n)}(0) = 0$ for some $n \geq 0$. Then

$$\lim_{x \to 0+} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0+} \frac{p_n(x)}{x^{2n+1}} e^{-1/x} = 0,$$

so $f^{(n+1)}(0) = 0$. ■

Exercise 6.6.4 a) For $f(x) = e^x$ and $x_0 = 0$, show that

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Proof: ■

b) For $f(x) = \sin x$ and $x_0 = 0$, show that

$$P_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1!}.$$

Proof: ■

c) For $f(x) = \log(x+1)$ and $x_0 = 0$, show that

$$P_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k}.$$

Proof: Just compute $f^{(n)}(0)$ in all cases. ■