MT804 Analysis Homework IV

Solutions

November 16, 2008

7.1.3, 7.1.5, 7.1.6 , 7.7.4, 7.7.5, 7.7.6, 7.7.11.

Exercise 7.1.3 Show that if the Riemann integral of f exists then it is unique.

Proof: Failure of uniqueness would mean there are two real numbers $I < J$ such that for all $\epsilon > 0$ there exist partitions P_ϵ, Q_ϵ with the properties that for any tagged partitions $\dot{P}, \dot{Q}$ with $P_\epsilon \subset P$ and $Q_\epsilon \subset Q$, we have $|S(f, \dot{P}) - I| < \epsilon$ and $|S(f, \dot{Q}) - J| < \epsilon$.

Let $\epsilon = (J - I)/2$ and take $P = P_\epsilon \cup Q_\epsilon = Q$, with arbitrary tags. Then

$$J - I = J - S(f, \dot{P}) + S(f, \dot{P}) - I \leq |J - S(f, \dot{P})| + |S(f, \dot{P}) - I| < 2\epsilon = J - I,$$

a contradiction.■

Exercise 7.1.5 Give an example of an absolutely integrable function that is not integrable.

Answer: The function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q} \\ -1 & \text{if } x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

is not integrable for the same reason that Dirichlet's function is not integrable. However $|f|$ is a constant function, which is integrable.

Exercise 7.1.6 Let f be integrable on $[a, b]$. Show that

$$\int_a^b f(x)\, dx = \int_{a+c}^{b+c} f(x - c)\, dx = \int_{-b}^{-a} f(-x)\, dx.$$

Proof: For the first one, let $f_c(x) = f(x-c)$. If $\dot{P} = ((x_j), (t_j))$ is any tagged partition of $[a, b]$, then $\dot{P}' = ((x_j + c), (t_j + c))$ is a tagged partition of $[a+c, b+c]$ and $S(f_c, \dot{P}') = S(f, \dot{P})$. Conversely, by subtracting c, we see that every tagged partition of $[a+c, b+c]$ is of the form $\dot{P}'$.

Let $\epsilon > 0$. Choose a partition P_ϵ satisfying the definition of integrability of f on $[a, b]$. Let $Q_\epsilon = P'_\epsilon$. if $\dot{Q}$ is any tagged partion of $[a+c, b+c]$ with $Q \supset Q_\epsilon$, then $\dot{Q} = \dot{P}'$ for a tagged partition $\dot{P}$ with $P \supset P_\epsilon$. By integrability on $[a, b]$, we have

$$|S(f_c, \dot{Q}) - \int_a^b f(x)\, dx| = |S(f, \dot{P}) - \int_a^b f(x)\, dx| < \epsilon.$$

By uniqueness of the integral, we have

$$\int_a^b f(x)\, dx = \int_{a+c}^{b+c} f(x-c)\, dx.$$

The second equality goes the same way. This time you multiply by -1 instead of adding c. ■

Exercise 7.7.4 Let $f : [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ is rational in lowest terms and } q > 0 \\ 0 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Show that $\int_0^1 f$ exists and equals zero.

Proof: Let $\epsilon > 0$. In earlier homework, we saw that the set of points in $[0, 1]$ where $f(x) \geq \epsilon/2\}$ is finite. Write

$$\{x \in [0, 1] :\ f(x) \geq \epsilon/2\} = \{y_1, \ldots, y_n\}.$$

Choose a partition $P = (x_i)_{i=0}^{2n}$ with

$$x_{2i-1} < y_i < x_{2i} < x_{2i+1} < y_{i+1} < x_{2i+2}$$

for $0 < i < n$, such that $x_{2i} - x_{2i-1} < \epsilon/2n$ for all i. Then $L(f, P) = 0$ and

$$U(f, P) < \sum_{i \text{ odd}} \frac{\epsilon}{2} \Delta x_i + \sum_{i \text{ even}} 1 \cdot \frac{\epsilon}{2n} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence f is integrable and $\int_0^1 f < \epsilon$ for arbitrary $\epsilon > 0$, hence $\int_0^1 f = 0$. ■

Exercise 7.7.5 Show that a bounded infinite set whose set of limit points is finite has content zero.

Proof: Let $S \subset [a, b]$ be a bounded infinite set and let $\{x_1, \ldots, x_n\}$ be the set of limit points of S. Let $\epsilon > 0$ and let $I_1, \ldots, I_n$ be intervals of length $\epsilon/2n$ such that $x_j \in I_j$ for each j. The set $S' = S - \cup I_j$ has no limit points, and is bounded. This implies that S' is finite.

Indeed, any y outside of S' is not a limit point of S' so there exists $\epsilon > 0$ such that $V_\epsilon(y) \cap S' = \varnothing$. This means S' is closed. Since it is also bounded, S' is compact. But for every $x \in S'$ there is $\delta > 0$ such that $V_\delta(x) \cap S' = \{x\}$, again since S' has no limit points. By compactness, S' must be finite: $S' = \{z_1, \cdots, z_m\}$.

Each z_i is contained in an interval I_{n+i} of width $\epsilon/2^{i+1}$. Then S is covered by the intervals $I_1, \ldots, I_{n+m}$ whose total length is

$$\sum_{i=1}^{n+m} \lambda(I_j) = n \cdot \frac{\epsilon}{2n} + \frac{\epsilon}{2} \sum_{i=1}^{m} 2^{-i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore S has content zero. ■

Exercise 7.7.6 Let f and g be functions on $[a, b]$ such that f is bounded and g is integrable. Assume also that the set $\{x \in [a, b] : \ f(x) \neq g(x)\}$ has content zero. Prove that f is integrable on $[a, b]$ and that $\int_a^b f = \int_a^b g$.

Proof: The function $h = f - g$ is bounded, say $|h| < M$, and is zero off of a subset $S \subset [a, b]$ with content zero. Let $\epsilon > 0$ and cover S with increasing disjoint closed intervals $I_1, \ldots, I_n$ such that $\sum \lambda(I_j) < \epsilon/2M$. Let $P = (x_j)_{j=0}^{2n}$ be the partition such that $[x_{2j-1}, x_{2j}] = I_j \cap [a, b]$. Then

$$U(h, P) - L(h, P) = \sum_{j=1}^{n} (M_{2j} - m_{2j})(x_{2j} - x_{2j-1}) < M \sum_{j=1}^{n} \lambda(I_j) < 2M \cdot \frac{\epsilon}{2M} = \epsilon.$$

Hence h is integrable, and if we choose any tags in P we have

$$|S(h, \dot{P})| < \epsilon/2.$$

It follows that $\int_a^b h = 0$. Therefore $f = g + h$ is integrable and

$$\int_a^b f = \int_a^b g + \int_a^b h = \int_a^b g.$$

Exercise 7.7.11 Define the average value of an integrable function $f : [a,b] \to \mathbb{R}$ to be the number

$$\operatorname{Av}(f) = \frac{1}{b-a} \int_a^b f.$$

a) Show that

$$\operatorname{Av}(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right).$$

b) Give an example of a non-integrable function for which the limit in part a) exists.

Proof: Let $\dot{P}_n$ be the tagged partition obtained by dividing $[a,b]$ into equal subintervals of width $(b-a)/n$, tagged with the right endpoint of each interval. Then

$$(b-a)\operatorname{Av}(f) = \int_a^b f = \lim_{n\to\infty} S(f, \dot{P}_n) = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) \cdot \frac{b-a}{n},$$

which proves the formula in a). If f is the Dirichlet function on $[0,1]$, which is discontinuous at every point of $[0,1]$ then f is not integrable. But f takes value=1 at every rational number, so the limit in part a) equals 1. ■