Math 814 Final Exam Solutions

1. Let f be a *nonconstant* entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

First suppose f is a polynomial. If $a \in \mathbb{C}$, the nonconstant polynomial $f(z) - a$ has a root, by the Fundamental Theorem of Algebra. Hence if f is a polynomial, we have the stronger result that $f(\mathbb{C}) = \mathbb{C}$. Assume from now on that f is entire, but not a polynomial. Then from class notes, we know that the function $g(z) = f(1/z)$ has an essential singularity at $z = 0$. By the Casorati-Weierstrass Theorem, for any $\delta > 0$ the image of $\{z : \ 0 < |z| < \delta\}$ under $g(z)$ is dense in $\mathbb{C}$. So we have the stronger result that for any $R > 0$, the image under $f(z)$ of $\{z : \ |z| > R\}$ is dense in $\mathbb{C}$.

Here is another proof, inspired by one of yours. If $f(\mathbb{C})$ is not dense in $\mathbb{C}$, then there is $w \in \mathbb{C}$ and $\epsilon > 0$ such that $|f(z) - w| > \epsilon$ for all $z \in \mathbb{C}$. Then $(f(z) - w)^{-1}$ is bounded entire, hence constant, which implies that f is constant.

2. Suppose f is analytic and one-to-one on a region U. Show that $f'(z) \neq 0$ for all $z \in U$. Discuss the converse.

Let $z_0 \in U$ and suppose $f'(z_0) = 0$. Then f vanishes to some order $m \geq 2$ at z_0. In class, we have seen that this means f is m-to-1 near z_0. This contradicts the assumption that f is $1 - 1$ on U.

The converse is false: The function $f(z) = e^z$ has $f'(z)$ never zero on $U = \mathbb{C}$, but $f(0) = f(2\pi i)$, so f is not $1 - 1$. However, the converse is *locally* true, in the sense that if $f'(z)$ is never zero on U, then around each $z_0 \in U$ there is an open disk $B(z_0, \delta)$ on which f is $1 - 1$. This follows from the same result used in the previous paragraph. For example, with $f(z) = e^z$, one can take $\delta = 2\pi$.

3. Suppose f and g are analytic on a region U, with the same zeros and multiplicities of zeros.

a) Show that f/g is analytic and nowhere zero on U.

The only possible points of nonanalyticity or vanishing are at one of the common zeros of f and g. Suppose f and g vanish to order $m > 0$ at some point $a \in U$. Then $f(z) = (z-a)^m f_1(z)$ and $g(z) = (z-a)^m g_1(z)$, where f_1 and g_1 are analytic on U and $f_1(a) \neq 0 \neq g_1(a)$. Hence $f/g = f_1/g_1$ is analytic nonvanishing at a.
b) Suppose $a \in U$, is a zero of f and g of multiplicity m. Show that

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{(m)}(a)}{g^{(m)}(a)}.$$

Use only facts which we have proved in class. L'Hôpital's rule is not one of them.

The power series expansion of $f(z)$ is

$$f(z) = \frac{f^{(m)}(a)}{m!}(z-a)^m + [\text{ higher powers of } (z-a)\ ],$$

so $f_1(a) = f^{(m)}(a)/m!$. Likewise, $g_1(a) = g^{(m)}(a)/m!$. Hence

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{f_1(a)}{g_1(a)} = \frac{f^{(m)}(a)}{g^{(m)}(a)}.$$

4. Let $U = \{z :\ 0 < |z| < 1\}$, and let γ be a closed path in U.

a) Let $f(z)$ be analytic on U and assume $\lim_{z\to 0} zf(z) = 1$. Show, by calculating it, that the integral $\int_\gamma f$ depends only on γ, not on f.

Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be the Laurent expansion of f on U. Then $zf(z)$ has the Laurent expansion

$$zf(z) = \sum_{n=-\infty}^{\infty} a_{n-1} z^n.$$

Since $zf(z) \to 1$, there are no negative powers of z in the series for $zf(z)$, and its constant term is $a_{-1} = 1$. Hence $\operatorname{Res}[f(z), 0] = a_{-1} = 1$, so

$$\int_\gamma f = 2\pi i \cdot n(\gamma, 0) \cdot 1$$

depends only on γ.

b) Find two functions g and h, both analytic on U, for which

$$\lim_{z\to 0} z^2 g(z) = 1 = \lim_{z\to 0} z^2 h(z),$$

yet $\int_\gamma g \neq \int_\gamma h$.

The functions $g(z)$ and $h(z)$ must have Laurent expansions $g(z) = \sum b_n z^n$, $h(z) = \sum c_n z^n$ such that $b_{-2} = c_{-2} = 1$, and no lower powers of z appear. We have to find g, h such that $b_{-1} \neq c_{-1}$. For example, if we take

$$g(z) = \frac{e^z}{z^2}, \qquad h(z) = \frac{\sin z}{z^3},$$

then $b_{-2} = 1 = c_{-2}$, yet $b_{-1} = 0$ and $c_{-1} = 1$.

5. In class we proved for $\lambda \in \mathbb{C} - \mathbb{Z}$, that

$$\frac{\pi^2}{\sin^2 \pi\lambda} = \sum_{k=-\infty}^{\infty} \frac{1}{(\lambda + k)^2}.$$

To do this, we integrated $\int_{\gamma_n} \pi \cot \pi z/(z + \lambda)^2$ over certain paths γ_n. Apply the same method and paths to $\int_{\gamma_n} \pi \cot \pi z/(z^2 - \lambda^2)$ to show that

$$\pi \cot \pi\lambda = \frac{1}{\lambda} + \sum_{k=1}^{\infty} \frac{2\lambda}{\lambda^2 - k^2}.$$

Choose n large enough so that $\pm\lambda$ are both inside γ_n. Since $z^2 - \lambda^2$ is again quadratic in z, the same estimates that we used for the first integral show that

$$\lim_{n\to\infty} \int_{\gamma_n} \frac{\pi \cot \pi z}{(z^2 - \lambda^2)} dz = 0.$$

The poles $\lambda, -\lambda, 1/k$ of the integrand inside γ_n are all simple, with residues

$$\operatorname{Res}\left[\frac{\pi \cot \pi z}{(z^2 - \lambda^2)}, \lambda\right] = \operatorname{Res}\left[\frac{\pi \cot \pi z}{(z^2 - \lambda^2)}, -\lambda\right] = \frac{\pi \cot \pi\lambda}{2\lambda},$$

$$\operatorname{Res}\left[\frac{\pi \cot \pi z}{(z^2 - \lambda^2)}, k\right] = \frac{1}{k^2 - \lambda^2}.$$

Hence

$$\sum_{k=-n}^{n} \frac{1}{k^2 - \lambda^2} + 2 \cdot \frac{\pi \cot \pi\lambda}{2\lambda} \to 0,$$

or

$$-\frac{1}{\lambda^2} + \sum_{k=1}^{n} \frac{2}{k^2 - \lambda^2} + \frac{\pi \cot \pi\lambda}{\lambda} \to 0,$$

as we wished to show. We could also write this as

$$\pi \cot(\pi\lambda) = \sum_{n=-\infty}^{\infty} \frac{\lambda}{\lambda^2 - k^2}. \tag{1}$$

Remarks: Replacing λ by z, we have now seen three similar-looking expansions of analytic functions on $\mathbb{C} - \mathbb{Z}$, with pole set $\mathbb{Z}$.

$$\pi \csc(\pi z) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{z+k},$$
$$\pi^2 \csc^2(\pi z) = \sum_{k=-\infty}^{\infty} \frac{1}{(z+k)^2}$$
$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{z}{z^2 - k^2}.$$

In contrast to the power series expansions, which involved Bernoulli numbers, these expansions are simple, make the periodicity of $\csc(\pi z)$ and $\cot(\pi z)$ evident.

This expansion of $\cot \pi z$ is particularly useful. We'll use it next semester to get the product formula for $\sin \pi z$ and thereby vindicate Euler's first, wild and crazy, way of computing $\sum 1/n^2$.

Instead of summing over $\mathbb{Z}$, one can sum over two-dimensional lattices in $\mathbb{C}$, leading to elliptic functions, which are doubly periodic, and are connected to many areas of mathematics, such as elliptic curves.

6. Let α be a fixed complex number. Define

$$(1+z)^\alpha = \exp(\alpha \text{Log}(1+z)),$$

where Log is the principal branch of the logarithm. Find the power series expansion of $(1+z)^\alpha$ about 0, and determine the radius of convergence.

Applying the chain rule to the right side, one proves the familar formula

$$\frac{d}{dz}(1+z)^\alpha = \alpha(1+z)^{\alpha-1}.$$

Then, for $f(z) = (1+z)^\alpha$, one proves by induction that

$$f^{(n)}(z) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+z)^\alpha,$$

which gives the famous Binomial Series

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n.$$

If α is an integer ≥ 0 then the coefficients are zero for $n > \alpha$, so $(1+z)^\alpha$ is a polynomial of degree n (as we knew!) with infinite radius of convergence. If α is not an integer ≥ 0 then the series converges up to the nearest point of non-analyticity of $\mathrm{Log}(1+z)$, which is $z=-1$. Hence the radius of convergence is ≥ 1. In fact the ratio test shows the radius of convergence is exactly one.

7. Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Calculate the following integrals.

a) $\int_\gamma \sin(\frac{1}{z})\ dz$

Since

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots,$$

we have $\mathrm{Res}[\sin(z^{-1}), 0] = 1$, so

$$\int_\gamma \sin(z^{-1})\ dz = 2\pi i.$$

b) $\int_\gamma \sin^2(z^{-1})\ dz$.

The quick answer is that the integral is zero, since the integrand is even, hence has no odd powers in its Laurent expansion, so its residue at 0 is zero. To see this explicitly, use the identity

$$\sin^2 z = \frac{1-\cos 2z}{2},$$

which we know holds for real z, hence for all $z \in \mathbb{C}$ since both sides are entire functions. This gives the power series

$$\sin^2 z = \frac{1}{2}\left[1 - 1 + \frac{(2z)^2}{2!} - \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} - + \cdots\right],$$

which gives the Laurent expansion

$$\sin^2\left(\frac{1}{z}\right) = \frac{2}{2!z^2} - \frac{2^3}{4!z^4} + \frac{2^5}{6!z^6} - + \cdots.$$

in which the coefficient of $1/z$ is zero.

8. Compute

$$\int_{-\infty}^{\infty} \frac{dz}{1+z+z^2+z^3+z^4}.$$

In class, we derived the formula

$$\int_{-\infty}^{\infty} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_{i=1}^{m} \frac{P(a_i)}{Q'(a_i)}, \tag{2}$$

where P and Q are polynomials with $\deg P \leq \deg Q - 2$, all poles of P/Q are simple and nonreal, and $a_1, \ldots, a_m$ are the poles of P/Q in the upper half-plane. In the present case there are two such poles:

$$a_1 = \alpha = e^{2\pi i/5}, \qquad a_2 = \alpha^2 = e^{4\pi i/5}.$$

Formula (2), with $P = 1$ and $Q(z) = 1 + z + z^2 + z^3 + z^4$, gives

$$\int_{-\infty}^{\infty} \frac{dz}{1+z+z^2+z^3+z^4} = 2\pi i \left[\frac{1}{1+2\alpha+3\alpha^2+4\alpha^3} + \frac{1}{1+2\alpha^2+3\alpha^4+4\alpha^6} \right].$$

Let's call this number $\mathcal{I}$, for "integral". There are various ways to simplify the above expression for $\mathcal{I}$. Mathematica can be coaxed to write it as

$$\mathcal{I} = \frac{\pi}{10\sqrt{2}} \left[(3+\sqrt{5})\sqrt{5-\sqrt{5}} + (3-\sqrt{5})\sqrt{5+\sqrt{5}} \right].$$

Using the values

$$\sin^2\left(\frac{2\pi}{5}\right) = \frac{5+\sqrt{5}}{8}, \qquad \sin^2\left(\frac{4\pi}{5}\right) = \frac{5-\sqrt{5}}{8},$$

$$\cos^2\left(\frac{2\pi}{5}\right) = \frac{3-\sqrt{5}}{8}, \qquad \cos^2\left(\frac{4\pi}{5}\right) = \frac{3+\sqrt{5}}{8},$$

we can also write it as

$$\mathcal{I} = \frac{4\pi}{5} \sin\left(\frac{4\pi}{5}\right) \left[2\cos^2\left(\frac{4\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) \right].$$

This simplifies even more, because $2\cos(2\pi/5)$ satisfies the equation $x^2 = 1 - x$, and this implies that

$$2\cos\left(\frac{2\pi}{5}\right) \left[2\cos^2\left(\frac{4\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) \right] = 1.$$

So in fact,

$$\mathcal{I} = \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}\right) = 2\pi \cdot \frac{\sqrt{5+\sqrt{5}}}{5\sqrt{2}}.$$

In hindsight, we could have arrived at this more easily if we had written the integrand as

$$\frac{z-1}{z^5-1}.$$

Then, with $P = z - 1$ and $Q = z^5 - 1$ and $\alpha = e^{2\pi i/5}$, formula (2) gives

$$\mathcal{I} = 2\pi i \cdot \frac{\alpha - 1}{5\alpha^4} + \frac{\alpha^2 - 1}{5\alpha^8} = \frac{2\pi i}{5}(\alpha^2 - \alpha + \alpha^{-1} - \alpha^2) = \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}\right),$$

since $\alpha^5 = 1$. The same method shows that

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x + \cdots + x^{2m}} = \frac{4\pi}{2m+1}\sum_k \sin\left(\frac{2k\pi}{2m+1}\right),$$

where the sum runs through all odd integers k such that $1 \leq k \leq m$.