Math 814 Exam 1
Due: Nov 7, 2007

You may consult books, homework solutions or the instructor, but no one else. All work must be your own.

1. Let $f(z) = z^2+1$. Find equations for and sketch the curves obtained as images of horizontal and vertical lines under $f(z)$.

We have $f = u+iv$ where $u = x^2-y^2+1$ and $v = 2xy$. First, $f(\mathbb{R}) = [1,\infty)$ and $f(i\mathbb{R}) = (-\infty, 1]$. If $x = a \neq 0$ is constant we have

$$u = 1 + a^2 - y^2, \quad v = 2ay,$$

so

$$u = 1 + a^2 - \frac{v^2}{4a^2}.$$

In the (u, v) plane, this is a left-opening parabola with vertex $(1 + a^2, 0)$. As $|a|$ increases, the vertex moves to the right and the parabola opens wider, becoming more like a straight line.

If $y = b \neq 0$ is constant we get

$$u = 1 - b^2 + \frac{v^2}{4b^2}.$$

In the (u, v) plane, this is a right-opening parabola with vertex $(1 - b^2, 0)$. As $|b|$ increases, the vertex moves to the left and the parabola opens wider, becoming more like a straight line. Each of these parabolas is perpendicular to all of the previous ones.

2. Repeat Problem 1 for $f(z) = \cos z$.

We have

$$f = \cos x \cosh y - i \sin x \sinh y,$$

so $u = \cos x \cosh y$, $v = -\sin x \sinh y$. There are three exceptional lines:

$$\begin{aligned} f(0 + i\mathbb{R}) &= \cosh \mathbb{R} = [1, \infty) \\ f(\tfrac{1}{2}\pi + i\mathbb{R}) &= -i \sinh \mathbb{R} = i\mathbb{R} \\ f(\mathbb{R} + 0i) &= \cos \mathbb{R} = [-1, 1]. \end{aligned}$$

Now for $x = a \notin \{0, \frac{1}{2}\pi\}$ we have

$$\left(\frac{u}{\cos a}\right)^2 - \left(\frac{v}{\sin a}\right)^2 = 1.$$

This is a hyperbola whose asympotes are $u = \pm(\tan a)v$, with vertices at $\pm\cos a$. As a moves from 0 to $\frac{1}{2}\pi$ the line $[1, \infty)$ opens into these hyperbolas which continue to open until becoming the imaginary axis.

For $y = b \neq 0$ we have

$$\left(\frac{u}{\cosh b}\right)^2 + \left(\frac{v}{\sinh b}\right)^2 = 1.$$

This is an ellipse with major axis $\cosh b$ and minor axis $|\sinh b|$ and foci at $\pm c$, where $c^2 = \cosh^2 b - \sinh^2 b = 1$. So all these ellipses have the same foci, namely ± 1. As b increases, $\sinh b / \cosh b \to 1$, so the ellipses become more and more circular. Each of them is perpendicular to all of the hyperbolas above.

3. Let $\mathfrak{H} = \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$ and let

$$M(z) = \frac{z - i}{z + i}.$$

(a) Find $M(\mathfrak{H})$.

(b) Find $M(\mathbb{R})$.

(c) Use the answer to (b) to find all solutions of $a^2 + b^2 = c^2$ in integers a, b, c.

It is probably easier to find $M(\mathbb{R})$ first. This is either a line or a circle. Since $M(\infty) = 1$ and $M(-i) = \infty$ and M is one-to-one, $M(\mathbb{R})$ cannot approach ∞, so $M(\mathbb{R})$ is the unique circle containing the three points

$$M(\infty) = 1, \quad M(0) = -1, \quad M(1) = \frac{1 - i}{1 + i} = -i.$$

Hence $M(\mathbb{R})$ is the circle $C := \{z : |z| = 1\}$.

Since $\mathbb{R}$ is the boundary of $\mathfrak{H}$, the circle C is the boundary of $M(\mathfrak{H})$, so $M(\mathfrak{H})$ is either the interior or exterior of C. Since $M(i) = 0$, it is the interior of C.

For the last part, finding a triple (a, b, c) is equivalent to finding the rational point

$$\frac{a}{c} + \frac{b}{c} i \in C.$$

If $t \in \mathbb{Q}$ then

$$M(t) = \frac{t - i}{t + i} = \frac{t^2 - 1 - 2it}{t^2 + 1},$$

which is a rational point on C. Given any rational point $z = x + iy \in C$ other than 1, you can check that we have

$$M^{-1}(z) = i\frac{1+z}{1-z} = \frac{y}{x-1} \in \mathbb{Q}.$$

Hence

$$M\left(\frac{y}{x-1}\right) = z,$$

so $M(\mathbb{Q})$ is precisely the set of all rational points on C. To make this more explicit, let $t = n/m$, with $n, m \in \mathbb{N}$. Then

$$M(t) = \frac{n^2 - m^2 - 2nmi}{n^2 + m^2},$$

so all Pythagorean triples are obtained in the form

$$a = n^2 - m^2, \qquad b = 2nm, \qquad c = n^2 + m^2,$$

for $m, n \in \mathbb{N}$.

4. Let w be any nonzero complex number. Find a sequence (z_n) in $\mathbb{C}$ such that $\lim z_n = 0$ and $e^{1/z_n} = w$ for all n.

Write $w = re^{i\theta}$ for $\theta \in [0, 2\pi)$. Then any branch of the logarithm has value

$$\log w = \log r + i(\theta + 2n\pi),$$

for some $n \in \mathbb{Z}$. Taking

$$z_n = \frac{1}{\log r + i(\theta + 2n\pi)}$$

gives the desired sequence.

5. Let $f(z)$ be an entire function such that

$$f(\mathbb{R}) = \mathbb{R} \qquad \text{and} \qquad f(i\mathbb{R}) = i\mathbb{R}.$$

Prove that $f(z)$ is an odd function. (Hint: Power series.)

Since f is entire, its 0-centered power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

converges for all z. If $z = x \in \mathbb{R}$, we have

$$\sum_{n=0}^{\infty} \bar{a}_n x^n = \overline{f(x)} = f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Hence $\bar{a}_n = a_n$ for all n, so all a_n are real. Next,

$$\sum_{n=0}^{\infty} a_n (ix)^n = f(ix) \in i\mathbb{R}.$$

The sum of the even terms in the series are real, so they must vanish. Hence the series for f consists only of odd powers of z, so f is odd.

6. Let $f(z)$ be an entire function and suppose there are constants M and $R > 0$ such that $|f(z)| \leq M|z|^n$ for all $z > R$. Show that $f(z)$ is a polynomial of degree at most n.

Imitate the proof of the Liouville theorem, but now using the fancier version of the Cauchy integral formula. Fix $z \in \mathbb{C}$ and let γ be a circle centered at z. Then we have

$$f^{(n+1)}(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{n+2}}\, dz.$$

Take $\gamma = z + \rho e^{it}$ for ρ large. We have

$$|z + \rho e^{it}| \leq |z| + \rho,$$

so

$$|f^{(n+1)}(z)| \leq \frac{1}{2\pi} \cdot 2\pi\rho \cdot \frac{M(|z|+\rho)^n}{\rho^{n+2}} = \frac{M(|z|+\rho)^n}{\rho^{n+1}}.$$

Since M and z are fixed, this can be made arbitrarily small by taking ρ large. Hence $f^{(n+1)}(z) = 0$. Since z was arbitrary, this means $f^{(n+1)} = 0$. Hence the power series of f at 0 terminates and f is a polynomial.