Math 814 HW 1

September 19, 2007

p.4, Exercise 1. *Prove that* $||z| - |w|| \leq |z - w|$ *and give conditions for equality.*

It is equivalent to prove that $||z| - |w||^2 \leq |z - w|^2$, which, after multiplying out and cancelling, amounts to proving that

$$-2|z||w| \leq -(z\bar{w} + \bar{z}w).$$

(Note that $z\bar{w} + \bar{z}w$ equals its own complex conjugate, hence is real.) Changing the signs and squaring, we have to prove that

$$4|z|^2|w|^2 \geq (z\bar{w})^2 + 2|z||w| + (\bar{z}w)^2,$$

i.e., that

$$(z\bar{w})^2 - 2|z||w| + (\bar{z}w)^2 \leq 0. \tag{1}$$

But

$$(z\bar{w})^2 - 2|z||w| + (\bar{z}w)^2 = (z\bar{w} - \bar{z}w)^2$$

and since $z\bar{w} - \bar{z}w$ equals its negative complex conjugate, we have $(z\bar{w} - \bar{z}w)^2$ real and ≤ 0. Hence (1) holds, which was equivalent the desired inequality.

We have equality iff $z\bar{w} - \bar{z}w = 0$, iff

$$\frac{z}{\bar{z}} = \frac{w}{\bar{w}}.$$

Writing $z = |z|e^{i\alpha}$ and $w = e^{i\beta}$ with $\alpha, \beta \in [0, 2\pi)$, this means that

$$e^{2i(\alpha - \beta)} = 1$$

so that $\alpha - \beta = 0$ or $\alpha - \beta = \pm\pi$. Hence we have equality iff z and w are on the same line through 0. ■

p.4, Exercise 2. *Let $z_1, \dots, z_n$ be nonzero complex numbers, with $n \geq 2$. Show that*

$$|z_1 + \cdots + z_n| = |z_1| + \cdots + |z_n|$$

if and only if all the ratios z_k/z_ℓ are real and nonnegative.

One direction is easy. If all the ratios are ≥ 0, we have $z_k = t_k z_1$ for each k, where $t_k \geq 0$. Let $t = t_1 + t_2 + \cdots + t_n$. Then

$$|z_1 + \cdots + z_n| = t|z_1| = t_1|z_1| + t_2|z_1| + \cdots + t_n|z_1| = |z_1| + \cdots + |z_n|.$$

For the other direction, first assume $n = 2$. Let $w = z_1/z_2$ and write $w = x + iy$. We have $|z_1 + z_2| = |z_1| + |z_2|$ iff $|1 + w| = 1 + |w|$. Squaring, this implies $y = 0$, so $w = x$. If $x < 0$ then $|1 + x| = 1 - x = |1 - x|$, which implies $x = 0$, a contradiction. Hence $w = x \geq 0$.

Now assume the result is true for $n - 1$ complex numbers and that

$$|z_1 + \cdots + z_n| = |z_1| + \cdots + |z_n|.$$

Let $\zeta_2 = z_2 + \cdots + z_n$. Then by the triangle inequality, we have

$$|z_1 + \cdots + z_n| = |z_1 + \zeta_2| \leq |z_1| + |\zeta_2| \leq |z_1| + \cdots + |z_n| = |z_1 + \cdots + z_n|.$$

Hence all the inequalities are equalities. By the result for $n-1$, we have $z_k/z_2 \geq 0$ for all $k \geq 2$. Hence $\zeta_2/z_2 \geq 0$. By the result for $n = 2$, we have $\zeta_2/z_1 \geq 0$. Hence we have $z_k/z_1 \geq 0$ for all k. ■

p.4, Exercise 3. *Let $a \in \mathbb{R}$ and fix $c > 0$. Describe the locus of points $z \in \mathbb{C}$ satisfying*

$$|z - a| - |z + a| = c.$$

By the triangle inequality there are no such points unless $c \leq |2a|$. In hindsight, it makes the equation come out nicer if we write $c = 2b$, where $0 < b \leq |a|$. Squaring, we then have

$$|z - a|^2 = 4b^2 + 4b|z + a| + |z + a|^2.$$

Writing $z = x + iy$, we get

$$(x - a)^2 + y^2 = 4b^2 + 4b|z + a| + (x + a)^2 + y^2,$$

or

$$-ax - b^2 = b|z + a|.$$

Squaring again, we get

$$a^2x^2 + 2ab^2x + b^4 = b^2[(x+a)^2 + y^2],$$

or

$$(a^2 - b^2)x^2 - b^2y^2 = b^2a^2 - b^4. \tag{2}$$

If $a^2 = b^2$ this is on the line $y = 0$, and the locus is the ray from a to ∞. If $a^2 > b^2$, equation (2) is a hyperbola. Since $b > 0$, the locus is the branch of the hyperbola farthest from a. If we set

$$\epsilon = \sqrt{\frac{a^2}{b^2} - 1},$$

then equation (2) becomes

$$\frac{x^2}{b^2} - \frac{y^2}{(\epsilon b)^2} = 1. \tag{3}$$

so the asymptotes are the lines $y = \pm\epsilon x$ and the foci are

$$\pm\sqrt{b^2 + (\epsilon b)^2} = \pm\sqrt{a^2 - b^2}.$$

If a is a general nonzero complex number we get hyperbolas meeting the axis from a to $-a$ perpendicularly, with the same equations, except a^2 is replaced by $|a|^2$. Note also that if $b = 0$, the locus is just the line bisecting the segment from a to $-a$ at its midpoint.

Some historical comments: This is an ancient problem, going back to Apollonius of Perga (262-190 BCE), who is famous for his books on Conics, which gave the first systematic analysis and names of hyperbolas, ellipses and parabolas. At that time, the complex plane or coordinates x, y were unknown. The problem would have been stated in words, something to the effect of:

> *Find the locus of points whose distances to two given points differ by a given constant.*

We have seen that the answer is (with some degenerate exceptions) a hyperbola. A similar problem is perhaps more familiar :

> *Find the locus of points whose distances to two given points sum to a given constant.*

Here, the minus sign in (3) becomes a plus sign, and the answer is an ellipse. There are analogous problems for the product and quotient, which we will consider later.

Lacking algebraic technology, Apollonius' treatment of these problems consists of many ingenious geometric Propositions which are difficult to follow, and which raised centuries of complaints from later mathematicians who struggled with them. One of the loudest whiners was Déscartes, who proposed what we now call Cartesian Coordinates precisely in order to understand Apollonius' Conics. Thanks to Déscartes, we now have an algebraic machine replacing Apollonius' ingenuity. Is there a price to pay for this advantage?

p.5, Exercise 5 *Let* $z = \cos(2\pi/n) + i\sin(2\pi/n)$*, where* n *is an integer* > 1. *Show that* $1 + z + \cdots + z^{n-1} = 0$.

Since $z^n = e^{2\pi i} = 1$ we have

$$0 = z^n - 1 = (z-1)(z^{n-1} + \cdots + z + 1).$$

Since $n > 1$ we have $z \neq 1$, so $z^{n-1} + \cdots + z + 1 = 0$.

Physical interpretation: Think of each power of z as a unit vector of force, based at the origin. The vectors are arranged equally around the circle, so the total force on the origin is zero.

p.5, Exercise 6 *Show that the function* $\varphi(t) = \cos(t) + i\sin(t)$ *is a homomorphism from the additive group of* $\mathbb{R}$ *to the multiplicative group* $T = \{z : \ |z| = 1\}$.

Use trigonometric identities for $\cos(t + t')$ and $\sin(t + t')$ to show that

$$\varphi(t + t') = \varphi(t) + \varphi(t').$$

Comments: The map φ is clearly surjective. The kernel of φ is $2\pi\mathbb{Z}$, so φ induces a group isomorphism $\mathbb{R}/2\pi\mathbb{Z} \simeq T$. Topologically, φ coils $\mathbb{R}$ upon the circle, as you would a garden hose.

p.10, Exercise 4. *Show that stereographic projection maps a circle on the unit sphere* S *to a circle or a line in* $\mathbb{C}$.

The circle is the intersection of S with a plane $\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = \ell$, normal to the unit vector $(\beta_1, \beta_2, \beta_3)$. By rotating around the north/south pole axis, we can assume that $\beta_1 = 0$. The stereographic projection sends a point (x_1, x_2, x_3) on S to the point

$$z = \frac{x_1 + ix_2}{1 - x_3} \tag{4}$$

in the complex plane. It suffices to find real constants c and r, with $r \geq 0$, such that

$$|z - ic| = r \tag{5}$$

whenever z comes from the circle on S, via (4). Squaring both sides of (5), we get

$$x_1^2 + [x_2 + c(x_3 - 1)]^2 = r^2(x_3 - 1)^2.$$

Since $x_1^2 + x_2^2 + x_3^2 = 1$, we can eliminate x_1 and write this as

$$1 - x_3^2 + 2cx_2(x_3 - 1) + c^2(x_3 - 1)^2 = r^2(x_3 - 1)^2. \tag{6}$$

Since $\beta_2 x_2 + \beta_3 x_3 = \ell$, we can multiply (6) by β_2 and then eliminate x_2:

$$\beta_2(1 - x_3^2) + 2c(\ell - \beta_3 x_3)(x_3 - 1) + \beta_2 c^2(x_3 - 1)^2 = \beta_2 r^2(x_3 - 1)^2. \tag{7}$$

We compare coefficients of $1, x_3, x_3^2$ and get three equations

$$\begin{aligned} \beta_2 - 2c\ell + \beta_2 c^2 &= \beta_2 r^2 \\ 2t(\ell + \beta_3) - 2\beta_2 c^2 &= -2\beta_2 r^2 \\ -\beta_2 - 2c\beta_3 + \beta_2 c^2 &= \beta_2 r^2. \end{aligned} \tag{8}$$

The middle equation is minus the sum of the other two, so we can forget it, and we need only find c, r satisfying

$$\begin{aligned} \beta_2 - 2c\ell + \beta_2 c^2 &= \beta_2 r^2 \\ -\beta_2 - 2c\beta_3 + \beta_2 c^2 &= \beta_2 r^2. \end{aligned} \tag{9}$$

Subtracting, solving for c and then solving for r, and using $\beta_2^2 + \beta_3^2 = 1$, we get

$$c = \frac{\beta_2}{\ell - \beta_3}, \qquad r = \frac{\sqrt{1 - \ell^2}}{\ell - \beta_3}. \tag{10}$$

The square-root is positive real: since $x_2, x_3 \in [-1, 1]$, we have $\ell = \beta_2 x_2 + \beta_3 x_3 \leq \max \beta_2, \beta_3 \leq 1$.

Reversing the process (Synthesis), one checks that if $\ell \neq \beta_3$ and c, r are as in (10), then (5) holds.

If $\ell = \beta_3$ then $\beta_2 \neq 0$ (lest our circle degenerate into the north pole), so the equations (10) are inconsistent and have no solution. We should expect this, since now our circle goes through $N = (0, 0, 1)$ so its stereographic image goes off to infinity. So we cannot solve (5), but the equation of the plane is now

$$\beta_2 x_2 = \beta_3(1 - x_3)$$

so the imaginary part of z is now

$$\frac{x_2}{1-x_3} = \frac{\beta_3}{\beta_2},$$

a constant, so z lies on a line parallel to the real axis.