Math 814 HW 2

September 29, 2007

p. 43: 1,4,6,13,15, p. 54 1, 3 (cos z only). $u(x,y) = x^3 - 3xy^2, u(x,y) = x/(x^2+y^2)$,

p.43, Exercise 1. Show that the function $f(z) = |z|^2 = x^2 + y^2$ has a derivative only at the origin.

On the region $U = \mathbb{C} - \{0\}$ we have $\bar{z} = f(z)/z$. If $f(z)$ were analytic at some $w \in U$ then $\bar{z}$, being the product of two functions analytic at w, would itself be analytic at w, which we know is false.

Consider now $w = 0$. Let $\epsilon > 0$. If $|z| < \epsilon$ then

$$\frac{|f(z) - f(0)|}{|z - 0|} = |z| < \epsilon,$$

so $f(z)$ is analytic at 0.

p.43, Exercise 4. Show that $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$.

There are two methods:

$$(\cos z)' = \frac{1}{2}(e^{iz} + e^{-iz})' = \frac{i}{2}(e^{iz} - e^{-iz}) = -\sin z,$$

and

$$(\cos z)' = \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)' = \sum_{n=0}^{\infty} \frac{(-1)^n 2n \cdot z^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n-1)!} = -\sin z.$$

It is similar for $(\sin z)'$.

p.43, Exercise 6. Describe the following sets:

$$\begin{aligned} \{z:\ e^z = i\} &= \left(2\mathbb{Z} + \tfrac{1}{2}\right)\pi i \\ \{z:\ e^z = -1\} &= (2\mathbb{Z} + 1)\pi i \\ \{z:\ e^z = -i\} &= \left(2\mathbb{Z} - \tfrac{1}{2}\right)\pi i \\ \{z:\ \cos z = 0\} &= \left(\mathbb{Z} + \tfrac{1}{2}\right)\pi \\ \{z:\ \sin z = 0\} &= \mathbb{Z}\pi. \end{aligned}$$

p.43, Exercise 13. Let $U = \mathbb{C} - \mathbb{R}_{\leq 0}$. Find all analytic functions $f(z)$ on U such that $z = (f(z))^n$.

Every branch of $\log z$ is of the form $\log z = \mathrm{Log}(z) + 2k\pi i$ for some $k \in \mathbb{Z}$, where $\mathrm{Log}(z)$ is the principal branch. Hence we have

$$z^{1/n} = e^{\log(z)/n} = e^{(\mathrm{Log}(z)+2k\pi i)/n} = e^{\mathrm{Log}(z)/n} \cdot e^{2k\pi i/n}.$$

The numbers $e^{2k\pi i/n}$ are precisely the n^{th} roots of unity; they depend only on the remainder of k modulo n. They are also the n distinct powers of $\zeta = e^{2\pi i/n}$. So the branches of $z^{1/n}$ on U are

$$\zeta^k \cdot e^{\mathrm{Log}(z)/n}, \qquad k = 0, 1, \ldots, n-1$$

and are all constant multiples of each other.

p.43, Exercise 15. Fix $r > 0$. Let A be the image under $e^{1/z}$ of the punctured disk $\{z:\ 0 < |z| < r\}$. Describe A.

The image of the punctured disk under $1/z$ is the infinite annulus

$$B = \{z : r^{-1} < |z|\}$$

and A is the image of B under e^z. I claim the image of A is $\mathbb{C} - \{0\}$, regardless of r. To see this, we have to prove that for $w \neq 0$, the equation

$$e^z = w$$

has a solution $z \in B$. Write $w = |w|e^{i\theta}$. We must find $z = x + iy$ such that $x^2 + y^2 > r^{-1}$ and

$$e^x e^{iy} = |w|e^{i\theta}.$$

So we want

$$e^x = |w|, \qquad y = \theta + 2k\pi,$$

for some $k \in \mathbb{Z}$. If we take $x = \log|w|$ and choose k large enough that

$$(\log|w|)^2 + (\theta + 2k\pi)^2 > r^{-1},$$

then $z = x + iy$ works.

Another way to see this is to write $z = \rho e^{i\theta}$ and consider the image of rays from the origin with fixed θ and $0 < \rho < r$. You get spirals that fill up the plane with 0 removed.

Additional Comment: The result shows that any arbitrarily small punctured neighborhood of 0 is sent by $e^{1/z}$ to the entire punctured plane. The point $z = 0$ is called an "essential singularity" of $e^{1/z}$ and this is an example of the Great Picard Theorem (p. 300 in the text).

p.54, Exercise 1. Find the image of $\{z : \operatorname{Re} z < 0, |\operatorname{Im} z| < \pi\}$ under e^z.

We have seen that the line $x = c$ is sent by e^z to a circle of radius e^c. Any segment of the line of length 2π is sent to the entire circle. The region is made out of such segments, so its image is the punctured disk

$$\{w : \ 0 < |w| < 1\}.$$

p.54, Exercise 3. Discuss the mapping properties of $\cos z$.

We have

$$\cos z = \cos x \cosh y - i \sin x \sinh y = u + iv.$$

First consider the image of the vertical lines $x = a \in [0, 2\pi)$.

If $a = 0$, the image is $[1, \infty)$.

If $a = \pi$, the image is $(-\infty, -1]$.

If $a = \pi/2$ or $a = 3\pi/2$, the image is $i\mathbb{R}$.

If a is none of the above, then

$$\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = \cosh^2 y - \sinh^2 y = 1.$$

This is a hyperbola with asymptotes $y = \pm \tan a$.

Next, consider the image of the horizontal lines $y = b$, for any $b \in \mathbb{R}$. If $b = 0$, the image is $[-1, 1]$. If $b \neq 0$, then

$$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = \cos^2 x + \sin^2 x = 1.$$

This is an ellipse with foci at ± 1 and eccentricity $\epsilon = \operatorname{sech} b$. The foci are the same for every b. For large b we have $\epsilon \sim 1$ and the ellipse is nearly a circle: the difference between the foci is negligible from far away. These ellipses for $y =$ constant are perpendicular to the hyperbolas coming from lines $x =$ constant.

So much for the images of horizontal and vertical lines. The *inverse* images of horizontal and vertical lines are the level curves of u and v. Here the picture is identical to that of $\sin z$ drawn in class, but shifted horizontally by $\pi/2$.

Extra Exercises. For $u(x, y) = x^3 - 3xy^2$ and $u(x, y) = x/(x^2 + y^2)$,

a) find a harmonic conjugate $v(x, y)$,

b) write the function $u(x, y) + iv(x, y)$ in terms of z,

c) sketch the level curves of $u(x, y)$ and $v(x, y)$.

The harmonic conjugates are

$$v = 3x^2y - y^3, \qquad \text{and} \qquad v = -\frac{y}{x^2 + y^2},$$

and we have

$$u + iv = z^3, \qquad \text{and} \qquad u + iv = \frac{1}{z},$$

respectively. One brute-force way to find these is to substitute

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}$$

into $u(x, y) + iv(x, y)$ and simplify until $\bar{z}$ disappears, which it will, as long as u, v satisfy the Cauchy-Riemann equations.

The level curves of $u = x^3 - 3xy^2$ are obtained as follows. The critical level curve is $u = 0$, which is three lines $x = 0, x = \pm\sqrt{3}y$, dividing the plane into six equal sectors. A level curve for $u = c \neq 0$ consists of three smooth approximations to the sharp corner in alternate sectors. The level curves of v are obtained by rotating the level curves of u by $\pi/2$. Remarkably, when you do this, the rotated curves are orthogonal to the original curves.

The level curves of $u(x, y) = x/(x^2 + y^2)$ are obtained as follows. First, since $(1/z)' = -1/z^2$ has no zeros, there are no critical points, except at $z = 0$. The level curve $u = 1/2c$ is the circle with radius $|c|$ and center $(c, 0)$. As c varies, we get the family of all circles tangent to $\mathbb{R}i$ at 0. The level curves of v are the circles tangent to $\mathbb{R}$ at 0, and are orthogonal to the previous circles.