Math 814 HW 3

October 16, 2007

p. 54: 9, 14, 18, 24, 25, 26

p.54, Exercise 9. If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions for T to preserve the unit circle.

T preserves the unit circle iff

$$|ae^{i\theta} + b| = |ce^{i\theta} + d|,$$

for all $\theta \in [0, 2\pi)$. Squaring both sides and multiplying out, we get

$$|a|^2 + |b|^2 + a\bar{b}e^{i\theta} + \bar{a}be^{-i\theta} = |c|^2 + |d|^2 + c\bar{d}e^{i\theta} + \bar{c}de^{-i\theta}.$$

Comparing coefficients, we get two equations:

$$|a|^2 + |b|^2 = |c|^2 + |d|^2, \qquad a\bar{b} = c\bar{d}.$$

Dividing the first by $|d|^2$ and using the second, we get

$$\frac{|c|^2}{|b|^2} + \frac{|b|^2}{|d|^2} = \frac{|c|^2}{|d|^2} + 1,$$

which can be written

$$\frac{|c|^2 - |b|^2}{|b|^2} = \frac{|c|^2 - |b|^2}{|d|^2}.$$

If the numerators are zero, this means

$$|c| = |b|, \quad \text{hence} \quad |a| = |d|.$$

If the numerators are nonzero, we have

$$|b| = |d|, \quad \text{hence} \quad |a| = |c|.$$

In the first case, there are u, v with $|u| = |v| = 1$ such that

$$c = ub, \qquad d = va.$$

Then we have

$$a\bar{b} = c\bar{d} = ub\bar{v}\bar{a},$$

so

$$\frac{ub}{\bar{b}} = \frac{va}{\bar{a}}.$$

This last number, call it λ, also has $|\lambda| = 1$, and we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}. \tag{1}$$

In the second case, there are u, v with $|u| = |v| = 1$ such that

$$d = ub, \qquad c = va.$$

This time, we have

$$a\bar{b} = c\bar{d} = va\bar{u}\bar{b},$$

so $v\bar{u} = 1$. But $\bar{u} = u^{-1}$, so $v = u$, and we have $ad - bc = aub - bua = 0$, which is illegal for a Möbius transformation. So the second case does not occur, and all such transformations T are of the form (1).

p.54, Exercise 14. Let G be the region between two circles inside one another, tangent at the point a. Map G conformally to the open unit disk.

The map $Tz = (z - a)^{-1}$ sends a to ∞, hence the circles go to lines ℓ, ℓ'. If ℓ and ℓ' are not parallel, they must meet at some point $z \in \mathbb{C}$. Then $T^{-1}z$ would be a point on both circles other than a. Since there is no such point, the lines ℓ, ℓ' are parallel. The original region G meets both circles, so T sends G to the region G_2 between the parallel lines ℓ and ℓ'.

Next $Rz = cz + d$ be a product of a translation, a rotation and a dilation sending G_2 to the region $G_3 = \{z \in \mathbb{C} : \ |\operatorname{Im}(z)| < \pi/2\}$. The function e^z sends the vertical line segment $\{z = k + iy, |y| < \pi/2\}$ to the semicircle of radius e^k in the half-plane $G_4 = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. Hence e^z maps G_3 conformally onto G_4.

Finally, the map

$$Sz = \frac{z - 1}{z + 1}$$

sends the imaginary axis $i\mathbb{R}$ to the circle $\{z : |z| = 1\}$ and the point $1 \in G_4$ to the the point 0. Hence S maps G_4 conformally onto the disk $\{z : |z| < 1\}$.

p.54, Exercise 18. Refer to the diagram on page 55. Since $M(ia) = 0$, the regions B, C, E, F which touch ia are sent, in some order, to the regions U, V, X, Y which touch 0. Since $M(ib) = \infty$, the regions B, E, which touch ib, are sent, in some order, to the regions U, X, which touch ∞. It follows that C, F go to V, Y, in some order. To determine which region goes to which, let x, y be small positive real numbers, so that the point $z = x + iy + ia$ belongs to E. Then the imaginary part of Mz is a positive number times $x(b - a)$, hence is positive. It follows that $ME = U$, and hence $MB = X$. Since B, C meet in a line, so does X, MC. It follows that $MC = Y$, so $MF = V$. Since A, B meet in a line, so does MA, X. Hence $MA = Z$ and finally $MD = W$. To summarize:

$$\begin{aligned} A &\mapsto Z \\ B &\mapsto X \\ C &\mapsto Y \\ D &\mapsto W \\ E &\mapsto U \\ F &\mapsto V. \end{aligned}$$

p.54, Exercise 24. Let T be a Möbius transformation with $T \neq I$. Show that if S is another Möbius transformation with the same fixed points as T, then S commutes with T.

By conjugating in the Möbius group, we may assume $T\infty = \infty$. This means $Tz = az + b$ for some $a, b \in \mathbb{C}$ with $a \neq 0$. Since S also fixes ∞, we have $Sz = cz + d$ for some $c, d \in \mathbb{C}$ with $c \neq 0$.

Suppose $a = 1$. Then T has ∞ as its only fixed-point. Conversely, since S also has ∞ as its only fixed point, it follows that $c = 1$. Hence S, T are translations, which commute with each other. (In fact $STz = z + b + d = TSz$.)

Suppose $a \neq 1$. Then T has a second fixed-point $z_0 = b/(1 - a)$. The translation $Rz = z - z_0$ fixes ∞ and sends $z_0 \mapsto 0$. Conjugating T and S by R, we may assume $z_0 = 0$. Hence T, S belong to the subgroup fixing 0 and ∞. Hence $Tz = az$ and $Sz = cz$, so S and T commute. ■

p.54, Exercise 25. Find all abelian subgroups of the group $\mathcal{M}$ of Möbius transformations.

Let $\mathcal{A}$ be an abelian subgroup of $\mathcal{M}$ and let $I \neq T \in \mathcal{A}$. As in the previous problem, conjugate so that $T\infty = \infty$.

Suppose that ∞ is the only fixed-point of T. For any $S \in \mathcal{A}$, we have

$$TS\infty = ST\infty = S\infty,$$

so $S\infty$ is also fixed by T. By uniqueness of the fixed-point, we have $S\infty = \infty$. If w is any fixed-point of S, then

$$STw = TSw = Tw.$$

Since S has at most two fixed-points, this implies that $Tw = w$ or $Tw = \infty$. Either way, we have $w = \infty$. Hence ∞ is the unique fixed-point of any nonidentity element in $\mathcal{A}$. It follows that $\mathcal{A}$ is contained in the subgroup of translations, which is isomorphic to $\mathbb{C}$.

Suppose now that T has another fixed-point besides ∞. As in the previous problem, we may assume this other fixed-point is 0, so that $Tz = az$. If $a \neq -1$, You can check directly that the only Möbius transformations commuting with T are of the form cz. Hence $\mathcal{A}$ is contained in the group of dilations and rotations, which is isomorphic to $\mathbb{C}^\times$ (the multiplicative group of nonzero complex numbers). Alternatively, the previous arguments show that if $I \neq S \in \mathcal{A}$ then $S\{0, \infty\} = \{0, \infty\}$. If S switches 0 and ∞, then $Sz = c/z$. But then we would have $STS^{-1} = T^{-1} \neq T$, since $a \neq -1$. So S must fix both $0, \infty$, hence (again) is of the form $Sz = cz$.

Finally, suppose $Tz = -z$. Then any $Sz = c/z$ commutes with T. Conjugating by $\sqrt{c}z$, we can assume $c = 1$. Hence T is contained in the abelian subgroup $\{z, -z, \frac{1}{z}, \frac{-1}{z}\}$, which is Klein's Viergruppe.

p.54, Exercise 26. Let $\varphi : GL_2(\mathbb{C}) \to \mathcal{M}$ be the map given by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}.$$

You can check directly that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{and} \quad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then $\varphi(AA') = \varphi(A) \circ \varphi(A')$. So φ is a homomorphism. If

$$\frac{az+b}{cz+d} = z \qquad \text{for all} \quad z,$$

then $az + b = z(cz + d)$. Comparing coefficients shows that $c = b = 0$ and $a = d$. So the kernel of φ is the subgroup $Z = \{aI :\ a \in \mathbb{C}^\times\}$ of $GL_2(\mathbb{C})$.

More conceptually, the homomorphism φ arises from the action of $GL_2(\mathbb{C})$ on lines in $\mathbb{C}^2$. If a matrix preserves every line, then every line is an eigenvector, which means that A is a scalar matrix. Note incidentally that the fixed points of $\varphi(A)$, which we studied in the previous problems, are exactly the eigenlines of A in $\mathbb{CP}^1$.

The restriction of φ to $SL_2(\mathbb{C})$ is still surjective, because for any $A \in GL_2(\mathbb{C})$, both A and $(\det A)^{-1/2}A$ have the same image under φ and the latter matrix has determinant $= 1$. The kernel of this restriction is then $Z \cap SL_2(\mathbb{C}) = \{\pm I\}$. So $SL_2(\mathbb{C})$ is a two-fold cover of $\mathcal{M}$.

There is a trap here, which, if you ever study algebraic groups, you will tumble into. Namely, not every *real* Möbius transformation (i.e., with $a, b, c, d \in \mathbb{R}$) comes from a matrix in $SL_2(\mathbb{R})$. I'll let you puzzle that out (hint: $-z$).