Math 814 HW 4

November 7, 2007

p. 74: 5, 6, 7, 9cd, 12, 13, 14.

Exercise 5. Give the power series expansion of $\operatorname{Log} z$ about $z = i$ and find its radius of convergence.

For any nonzero $a \in \mathbb{C}$, we have

$$\frac{1}{z} = \frac{1}{a} \cdot \frac{1}{1 + \frac{z-a}{a}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (z-a)^n,$$

with radius of convergence $|a|$. Take $a = i$, antidifferentiate, and remember that $i^2 = -1$, $\operatorname{Log} i = i\pi/2$. You get

$$\operatorname{Log} z = \frac{i\pi}{2} - \sum_{n=0}^{\infty} \frac{i^{n+1}}{n+1} (z-i)^{n+1} = \frac{i\pi}{2} - \sum_{n=1}^{\infty} \frac{(iz+1)^n}{n},$$

with radius of convergence $|i| = 1$.

Exercise 6. Give the power series expansion of $\sqrt{z}$ about $z = 1$ and find its radius of convergence.

There are two branches of $\sqrt{z}$, differing by a sign, which can be detected from the value ± 1 at $z = 1$. Choose the branch $f(z)$ such that $f(1) = 1$. For $n > 0$, we have (CORRECTED VERSION)

$$f^{(n)}(1) = (-1)^{n-1} \frac{(2n-2)!}{2^{2n-1}(n-1)!}.$$

(Note: This is better than writing

$$f^{(n)}(1) = (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n-3)}{2^n},$$

since the latter is ambiguous at $n = 1$.) We get

$$f(z) = 1+\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!n!}(z-1)^n = 1-2\sum_{n=1}^{\infty} \frac{1}{2n-1}\binom{2n-1}{n}\left(\frac{1-z}{4}\right)^n$$

and the radius of convergence R is the distance to the nearest nonanalytic point, which is $z = 0$, so $R = 1$.

Exercise 7. In problems 7,9, let $\gamma_0(t) = e^{it}$, for $t \in [0, 2\pi]$.

a)

$$\int_{\gamma_0} \frac{e^{iz}}{z^2}\, dz = 2\pi i \cdot f'(0),$$

where $f(z) = e^{iz}$. So the integral is $2\pi i \cdot i = -2\pi$.

b)

$$\int_{a+r\gamma_0} \frac{dz}{z-a} = 2\pi i.$$

c)

$$\int_{\gamma_0} \frac{\sin z}{z^3}\, dz = 2\pi i \cdot f''(0),$$

where $f(z) = \sin z$. Hence the integral is $-2\pi i \cdot \sin 0 = 0$.

d) The integral

$$\int_{1+\frac{1}{2}\gamma_0} \frac{\operatorname{Log} z}{z^n}\, dz$$

is zero, since $z^{-n} \operatorname{Log} z$ is analytic in a disk containing the path.

Exercise 9.

c) First,

$$\frac{1}{z^2+1} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right].$$

Both $(z \pm i)^{-1}$ integrate to $2\pi i$ around $2\gamma_0$. Hence

$$\int_{2\gamma_0} \frac{dz}{z^2+1} = \frac{1}{2i}[2\pi i - 2\pi i] = 0.$$

Alternatively, note that

$$M(z) = \frac{z-i}{z+i}$$

maps $\mathbb{C} - [-i, i]$ to $\mathbb{C} - \mathbb{R}_{\leq 0}$, so $\operatorname{Log} M(z)$ is analytic on the region $\mathbb{C} - [-i, i]$ containing $2\gamma_0$. Moreover,

$$(\operatorname{Log} M(z))' = \frac{1}{z-i} - \frac{1}{z+i}.$$

Hence the integral is zero.

d)

$$\int_{\gamma_0} \frac{\sin z}{z}\, dz = 2\pi i \cdot \sin(0) = 0.$$

Alternatively, note that $\sin z/z$ is entire, hence has zero integral over every closed path in $\mathbb{C}$.

Exercise 12. Since $\sec z$ is even and $\sec 0 = 1$, it follows that

$$\sec z = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} z^{2k},$$

where the radius of convergence is the distance from 0 to the nearest non-analytic point(s) of $f(z) = \sec z$, which is $\pi/2$, and $E_{2k} = f^{2k}(0)$.

Multiplying the series for $\sec z$ and $\cos z$, we get

$$1 = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} (-1)^{n-k} \frac{E_{2k}}{(2k)!(2n-2k)!} \right] z^{2n}.$$

Comparing coefficients of z^{2n} and multiplying by $(2n)!$, we get the recursive formula

$$\sum_{k=0}^{n} (-1)^{n-k} E_{2k} \binom{2n}{2k} = 0.$$

We have

$$E_0 = 1, \quad E_2 = 1, \quad E_4 = 5, \quad E_6 = 61, \quad E_8 = 1385.$$

Exercise 13. We have

$$\frac{e^z - 1}{z} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!},$$

with infinite radius of convergence. The series

$$f(z) = \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

has radius of convergence R equal to the distance from 0 to the nearest zero(s) of $e^z - 1$, which are $\pm 2\pi i$, so $R = 1$.

Multiplying these two series, we get

$$1 = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \frac{a_k}{k!(n-k+1)!} \right] z^n.$$

Comparing coefficients of z^n and multiplying by $(n+1)!$, we get the recursive formula

$$\sum_{k=0}^{n} a_k \binom{n+1}{k} = 0.$$

Taking $n = 1$ and using $a_0 = f(0) = 1$, we find that $a_1 = -\frac{1}{2}$. The function

$$\tilde{f}(z) = f(z) + \frac{z}{2} = \frac{z(e^z + 1)}{2(e^z - 1)} = 1 + \sum_{k=2}^{\infty} \frac{a_k}{k!} z^k$$

is even, so $a_k = 0$ for k odd, $k > 1$. Let $B_{2n} = (-1)^{n-1} a_{2n}$, so that

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} z^{2n}.$$

We have

$$B_2 = \frac{1}{6}, \quad B_4 = \frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{1}{30}, \quad B_{10} = \frac{5}{66}.$$

Exercise 14. Find the power series of $\tan z$ in terms of Bernoulli numbers. Replace z by $2iz$ in the function $\tilde{f}(z)$ of the previous problem. We get

$$\tilde{f}(2iz) = \frac{iz(e^{2iz} + 1)}{e^{2iz} - 1} = \frac{iz(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}} = z \cot z.$$

Replacing z by $2iz$ in the power series for $\tilde{f}(z)$, we get

$$z \cot z = 1 - \sum_{n=1}^{\infty} \frac{4^n B_{2n}}{(2n)!} z^{2n}.$$

Now,

$$\cot 2z = \frac{\cos 2z}{\sin 2z} = \frac{\cos^2 z - \sin^2 z}{2\sin z \cos z} = \frac{1}{2}(\cot z - \tan z),$$

so

$$\begin{aligned} z \tan z &= z \cot z - 2z \cot 2z \\ &= \left[1 - \sum_{n=1}^{\infty} \frac{4^n B_{2n}}{(2n)!} z^{2n}\right] - \left[1 - \sum_{n=1}^{\infty} \frac{4^{2n} B_{2n}}{(2n)!} z^{2n}\right] \\ &= \sum_{n=1}^{\infty} \frac{4^n(4^n-1)B_{2n}}{(2n)!} z^{2n}. \end{aligned}$$

Hence we get

$$\tan z = \sum_{n=1}^{\infty} \frac{4^n(4^n-1)B_{2n}}{(2n)!} z^{2n-1}.$$