Math 814 HW 5

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p. 87: 6, 7 p. 96: 8a, 10, 11 p. 110: 1bchi, 5,13.

p.87, no. 6. Let f be analytic on $D = B(0,1)$ and suppose $|f(z)| \leq 1$ on D. Show that $|f'(0)| \leq 1$.
Proof: Let $0 < r < 1$ and let $\gamma_r(t) = re^{it}$ for $0 \leq t \leq 2\pi$. By the Cauchy Integral Formula, we have

$$f'(0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w^2}\, dw,$$

so

$$|f'(0| \leq \frac{1}{2\pi} \cdot \frac{1}{r^2} \cdot 2\pi r = \frac{1}{r}.$$

Taking the limit as $r \to 1$, we have $|f'(0)| \leq 1$. ■

p.87, no. 7. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$ and let $n \in \mathbb{N}$. Find

$$\int_\gamma \left(\frac{z}{z-1}\right)^n dz.$$

Apply C.I.F. to $f(z) = z^n$, to get

$$\int_\gamma \left(\frac{z}{z-1}\right)^n dz = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(1) = 2n\pi i.$$

You can also do this without C.I.F., by computing directly:

$$\begin{aligned}\int_\gamma \left(\frac{z}{z-1}\right)^n dz &= \int_0^{2\pi} \left(\frac{1+e^{it}}{e^{it}}\right)^n \cdot ie^{it}\,dt \\ &= i\int_0^{2\pi} \left(1+e^{-it}\right)^n \cdot e^{it}\,dt \\ &= i\int_0^{2\pi} \left(1+ne^{-it}+\binom{n}{2}e^{-2it}+\cdots\right)\cdot e^{it}\,dt \\ &= 2n\pi i,\end{aligned}$$

since $\int_0^{2\pi} e^{kit}\,dt = 0$ for k a nonzero integer.

p.96, no. 8a. We must integrate $(z-a)^{-1}$ and $(z-b)^{-1}$ over the path γ, which can be written as a sum of six paths, two of which are closed and have a, b in their ∞-components, hence have zero integral and two pairs of non-closed paths. One pair starts at the leftmost crossing point, each goes around a in opposite directions, and they meet at the middle crossing point. The other one pair starts at the middle crossing point, each goes around b in opposite directions, and they meet at the rightmost crossing point.

Integrating over the paths around a is the same as integrating $(z-a)^{-1}$ over $\gamma_1 - \gamma_2$, where $\gamma_1(t) = a + re^{it}$ for $0 \le t \le \pi$ and $\gamma_2(t) = a + re^{it}$ for $\pi \le t \le 2\pi$, for some small $r > 0$. One computes

$$\int_{\gamma_1} \frac{dz}{z-a} = \pi i = \int_{\gamma_2} \frac{dz}{z-a},$$

hence $n(\gamma, a) = 0$. Similarly, $n(\gamma, b) = 0$.

p.96, no. 10. Compute $\int_\gamma (1+z^2)^{-1}$ for all closed paths not passing through $\pm i$.

$$\begin{aligned}\int_\gamma \frac{dz}{1+z^2} &= \frac{1}{2i}\int_\gamma \left(\frac{1}{z-i} - \frac{1}{z+i}\right) dz \\ &= \frac{1}{2i}\cdot 2\pi i \cdot (n(\gamma, i) - n(\gamma, -i)) \\ &= \pi \cdot (n(\gamma, i) - n(\gamma, -i)).\end{aligned}$$

p.96, no. 11. The Cauchy integral formula can be written

$$\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz = 2\pi i \cdot n(\gamma, a) \cdot \frac{f^{(n)}(a)}{n!}.$$

Since $(e^z - e^{-z})'''(0) = 2$, we have

$$\int_\gamma \frac{e^z - e^{-z}}{z^4} dz = \frac{2\pi i}{3} \cdot n(\gamma, 0),$$

giving the answers

a) $\frac{2\pi i}{3}$,

b) $\frac{4\pi i}{3}$,

c) $\frac{4\pi i}{3}$.

p.110, no. 1.

b) $\cos z/z = 1/z +$ (higher powers) has a simple pole at $z = 0$.

c) $f(z) = (\cos z - 1)/z = -z/2 +$(higher powers) has a removable singularity at $z = 0$ and $f(0) = 0$.

h) $1/(1 - e^z) = -1/z +$ (higher powers) has a simple pole at $z = 0$.

i) Since $\sin z/z$ is entire, the function $z \sin(1/z)$ has an essential singularity at $z = 0$. We consider the function $g(z) = \sin z/z$ for z large. We first invoke the Little Picard Theorem, which asserts that in any neighborhood $|z| > R$ of ∞, we have either $g(U) = \mathbb{C}$ or $g(U) = \mathbb{C} - \{w\}$, for some $w \in \mathbb{C}$. I claim that in this case it is the former. Assume there is such a w. Then since $g(\bar{z}) = \overline{g(z)}$, it follows that w is real. We have

$$g(x+iy) = u+iv = \frac{x \sin x \cosh y + y \cos x \sinh y}{x^2 + y^2} + i\frac{x \cos x \sinh y - y \sin x \cosh y}{x^2 + y^2}.$$

It is easy to see that $g(\mathbb{R}) = [-2/3\pi, 1]$ and $g(i\mathbb{R}) \supset [1, \infty]$. Hence this hypothetical w must lie in $(-\infty, -2/3\pi)$. Now, g is real on the curve

$$x \cos x \sinh y - y \sin x \cosh y = y(\frac{x \cos x \sinh y}{y} - \sin x \cosh y) = 0.$$

Since we already know $g(\mathbb{R})$, we set the factor in $(\dots)$ equal to zero, and get the curve

$$C: \; y \coth y = x \cot x.$$

This curve C meets the x-axis at the solutions of $\tan x = x$, which form a sequence $z_n, \; n \in \mathbb{Z}$, such that $z_n \to (n - \frac{1}{2})\pi$ as $|n| \to \infty$. On C, we have

$$g|_C = u|_C = \frac{\sin x \cosh y}{x} = \frac{\cos x \sinh y}{y}.$$

For n a large positive integer, let $x_n = 3\pi/4 - 2n\pi$ and let y_n satisfy $y_n \coth y_n = -x_n$. (Note that $y \coth y$ is unbounded, so such y_n exists.) Then since $\cot x_n = -1$, the point (x_n, y_n) lies on C. Let A_n be the arc on C from (x_n, y_n) to the x-axis and let $(z_n, 0)$ be the point where A_n meets the x-axis. Since A_n is connected, the set $g(A_n)$ is an interval. We have

$$u(x_n, y_n) = \frac{\cosh y_n}{\sqrt{2}x_n} = -\frac{\sinh y_n}{\sqrt{2}y_n}, \quad \text{and} \quad u(z_n, 0) = \cos z_n \approx \cos(\pi/2 + n\pi) = 0.$$

Hence $u(A_n)$ contains the interval $[-\sinh y_n/\sqrt{2}y_n, \epsilon]$, where $\epsilon > 0$ is small. For large enough n, this will overlap with our previously obtained interval $[-2/3\pi, \infty)$. So for large n we now have $[-\sinh y_n/\sqrt{2}y_n, \infty)$ in the image of g. Since $y_n \to \infty$, we have $-y_n^{-1} \sinh y_n \to -\infty$, so the entire negative real axis is covered. Whew!

p.110, no. 5. Let $a_n = \frac{\pi}{2} + n\pi$. Since $\tan z = \frac{\sin z}{\cos z}$ and $(\cos z)' = -\sin z$, it follows that $\tan z$ has a simple pole at each a_n, with residue -1. Hence the singular part of $\tan z$ at a_n is $-1/(z - a_n)$.

p.110, no. 13. a) If $f(z)$ is entire and $\lim_{z\to\infty} f(z)$ exists and is finite, then f is bounded, so f is constant, by Liouville's theorem.

b) If $f(z)$ is entire and has a pole of order m at ∞, then $f(1/z)$ has a pole of order m at 0. Hence $f(z) = z^{-m}g(z)$, where $g(z)$ is analytic and nonzero at 0, so $f(z) = z^m g(1/z)$ and $g(1/z)$ is bounded, for $|z|$ large, say $|g| \leq M$. Then $|f(z)| \leq M|z|^m$ for $|z|$ large, so f is a polynomial of degree m, by the extension of Liouville from the first exam.

c) A rational function $P(z)/Q(z)$, with $P, Q \in \mathbb{C}[z]$, is bounded for $|z|$ large if and only if $\deg P \leq \deg Q$.

d) A rational function $P(z)/Q(z)$ has a pole of order m at ∞ iff $P(z)/Q(z) = z^m g(z)$, where $g(z)$ is a rational function bounded near ∞. This means $\deg P = m + \deg Q$.