MT815 Complex Variables Homework I

Your Name

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p.176: 1,3,4 p.185: 2,7,8

p. 176, Exercise 1. Show that

$$\cos \pi z = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2} \right].$$

We proceed as with $\sin \pi z$. The zeros of $\cos \pi z$ are...

p. 176, Exercise 3. Find a factorization for the function

$$f(z) = \cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right).$$

We proceed as before. The zeros of $f(z)$ are...

p. 176, Exercise 4. Prove Wallis' formula:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

This is one of the most important formulas in the development of calculus. You don't need complex variables to prove it. See my course notes for MT105.

Extra Exercise 1. Use Euler's method with the product formula for $\sin \pi z$ to compute $\sum_{n=1}^{\infty} n^{-4}$. What happens with $\sum n^{-3}$?

p. 185, Exercise 2. Show that $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$. Deduce from this that $\Gamma(1/2) = \sqrt{\pi}$. What does this have to do with Wallis' formula?

p. 185, Exercise 7. Show that

$$\int_0^\infty \sin(t^2)\, dt = \int_0^\infty \cos(t^2)\, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

For more about these integrals, see

`http://en.wikipedia.org/wiki/Fresnel_integral.`

p. 185, Exercise 8. Show that the volume of the unit ball B_n in $\mathbf{R}^n$, for $n \geq 1$, is given by

$$\operatorname{vol}(B_n) = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)}.$$

Simplify this when $n = 2k$ is even. Check the result for $n = 1, 2, 3$.