MT815 Complex Variables Homework II

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Exercise 1. Show that for $a > 0$ we have

$$\int_0^\infty e^{-(t^a)}\,dt = \Gamma\left(1 + \frac{1}{a}\right).$$

Make the substitution $u = t^a$ and get

$$\int_0^\infty e^{-(t^a)}\,dt = \int_0^\infty e^{-u}\frac{u^{1/a}}{a}\frac{du}{u} = \frac{1}{a}\Gamma\left(\frac{1}{a}\right) = \Gamma\left(1 + \frac{1}{a}\right).$$

Exercise 2. Verify the following calculations that we needed in the proof of the integral formula for $\Gamma(z)$:

(a)
$$\int_0^t \frac{u}{n}\left(1 - \frac{u}{n}\right)^{n-1} e^u\,du = 1 - e^t\left(1 - \frac{t}{n}\right)^n.$$

(b)
$$\int_0^1 y^{z-1}(1-y)^n\,dy = \frac{n!}{z(z+1)\cdots(z+n)}.$$

(You cannot use the Beta integral formula here, since we needed the integral formula to derive it!)

For (a), note that the left side is the unique function $f(t)$ such that

$$f'(t) = \frac{t}{n}\left(1 - \frac{t}{n}\right)^{n-1} e^t, \qquad f(0) = 0.$$

On the other hand, the right side vanishes at zero and its derivative is

$$-e^t\left(1-\frac{t}{n}\right)^n - ne^t\cdot\frac{-1}{n}\cdot\left(1-\frac{t}{n}\right)^{n-1} = \frac{t}{n}\left(1-\frac{t}{n}\right)^{n-1}e^t.$$

For (b), denote the integral by $F_n(z)$ and use integration by parts (and the fact that $\Re z > 0$) to get

$$F_n(z) = \frac{n}{z}\cdot F_{n-1}(z+1).$$

The result follows from induction on n.

Exercise 3. Prove Euler's formula:

$$\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}},$$

using the following steps.

(a)

$$\prod_{k=1}^{n-1} e^{ik\pi/n} = i^{n-1}.$$

(b)

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = n\cdot 2^{1-n}.$$

(Use $2i\sin z = e^{iz}(1-e^{-2iz})$.)

(c)

$$\left(\prod_{k=1}^{n-1}\Gamma\left(k/n\right)\right)^2 = \prod_{k=1}^{n-1}\frac{\pi}{\sin\left(k\pi/n\right)}.$$

(d) Deduce Euler's formula above.

Since

$$\sum_{k=1}^{n-1}\frac{ik\pi}{n} = \frac{i\pi}{n}\sum_{k=1}^{n-1}k = \frac{i\pi}{n}\cdot\frac{n(n-1)}{2} = \frac{i\pi(n-1)}{2},$$

we have

$$\prod_{k=1}^{n-1} e^{ik\pi/n} = (e^{\pi i/2})^{n-1} = i^{n-1}.$$

Now

$$\prod_{k=1}^{n-1} \frac{\pi}{\sin(k\pi/n)} = \prod_{k=1}^{n-1} \frac{e^{k\pi i/n}(1-e^{-2k\pi i/n})}{2i} = \frac{i^{n-1}}{(2i)^{n-1}} \prod_{k=1}^{n-1}(1-\zeta^k),$$

where $\zeta = e^{-2\pi i/n}$. These powers of ζ run through all roots of $z^n - 1 = 0$ except $z = 1$, so

$$\prod_{k=1}^{n-1}(z-\zeta^k) = \frac{z^n-1}{z-1},$$

which becomes n when $z = 1$. Hence

$$\prod_{k=1}^{n-1} \frac{\pi}{\sin(k\pi/n)} = \frac{n}{2^{n-1}}.$$

Finally, the functional equation $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ gives

$$\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)^2 = \prod_{k=1}^{n-1}\left[\Gamma\left(\frac{k}{n}\right)\Gamma\left(\frac{n-k}{n}\right)\right] = \prod_{k=1}^{n-1} \frac{\pi}{\sin(k\pi/n)}.$$

Exercise 4. This exercise relates the value $\Gamma(1/4)$ to elliptic integrals.

(a) Let $k \geq 0$. Compute the integral

$$I_k = \int_0^1 \frac{t^k}{\sqrt{1-t^4}}\,dt$$

in terms of π and $\Gamma(1/4)$.

(b) Deduce from the previous exercise that

$$I_0 \cdot I_2 = \frac{\pi}{4}.$$

(c) In Calculus, we learn that the arclength L of the ellipse

$$x^2 + 2y^2 = 2$$

is given by the elliptic integral

$$L = 4 \int_0^{\pi/2} \sqrt{1 - \frac{1}{2}\sin^2\theta}\, d\theta.$$

Rewrite this integral in terms of $\cos\theta$, make a substitution, and show that

$$L = 2\sqrt{2}(I_0 + I_2).$$

Express this elliptic arclength is in terms of the circular arclength π and the value $\Gamma(\frac{1}{4})$.

(d) Deduce from the previous two exercises that I_0 and I_2 are the roots of the polynomial

$$4x^2 - \sqrt{2}Lx + \pi = 0.$$

Make the substitution $u = t^4$ to get

$$I_k = \frac{1}{4}\int_0^1 u^{(k-3)/4}(1-u)^{-1/2}\, du = \frac{\Gamma\left(\frac{k+1}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{4\cdot\Gamma\left(\frac{k+3}{4}\right)} = \frac{\Gamma\left(\frac{k+1}{4}\right)\cdot\sqrt{\pi}}{4\cdot\Gamma\left(\frac{k+3}{4}\right)}.$$

Both Γ values can be reduced to $\Gamma(1/4)$. For example, we have

$$\Gamma(1/4)\cdot\Gamma(3/4) = \pi\csc(\pi/4) = \pi\sqrt{2},$$

so

$$I_0 = \frac{\Gamma(1/4)\sqrt{\pi}}{4\Gamma(3/4)} = \frac{\Gamma(1/4)^2}{4\sqrt{2\pi}}$$

and

$$I_2 = \frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(5/4)} = \frac{\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)} = \frac{\pi\sqrt{2\pi}}{\Gamma(1/4)^2},$$

so

$$I_0\cdot I_2 = \frac{\pi}{4}.$$

Turning to the ellipse, writing $\sin^2\theta = 1 - \cos^2\theta$ gives

$$L = 2\sqrt{2}\int_0^{\pi/2} \sqrt{1+\cos^2\theta}d\theta.$$

The substitution $u = \cos\theta$ gives

$$L = 2\sqrt{2}\int_0^1 \frac{\sqrt{1+u^2}}{\sqrt{1-u^2}}du = 2\sqrt{2}\int_0^1 \frac{1+u^2}{\sqrt{1-u^4}}du = 2\sqrt{2}(I_0+I_2).$$

Using the calculations of I_0 and I_2 from problem 4, we find that the arclength of the ellipse is given by

$$L = 2\sqrt{2}\left[\frac{\Gamma(1/4)^2}{4\sqrt{2\pi}} + \frac{\pi\sqrt{2\pi}}{\Gamma(1/4)^2}\right] = \sqrt{\pi}\left[\frac{\Gamma(1/4)^2}{2\pi} + \frac{4\pi}{\Gamma(1/4)^2}\right].$$

Now I_0 and I_2 are the zeros of the polynomial

$$(x - I_0)(x - I_2) = x^2 - (I_0 + I_2)x + I_0 I_2,$$

which leads to the equation $4x^2 - \sqrt{2}Lx + \pi = 0$. It is perhaps nicer to say that $2\sqrt{2}I_0$ and $2\sqrt{2}I_2$ are the zeros of

$$x^2 - Lx + 2\pi,$$

a polynomial whose coefficients are the arclengths of the ellipse and the circle.

Exercise 5. In analogy with the basic formula

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2},$$

where π is the circular constant, we write

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\varpi}{2},$$

where ϖ (“varpi” in TeX) is the *Lemniscatic Constant*. The lemniscate has equation $|z^2 - \frac{1}{2}| = \frac{1}{2}$, or when squared, $(x^2 + y^2)^2 = x^2 - y^2$.

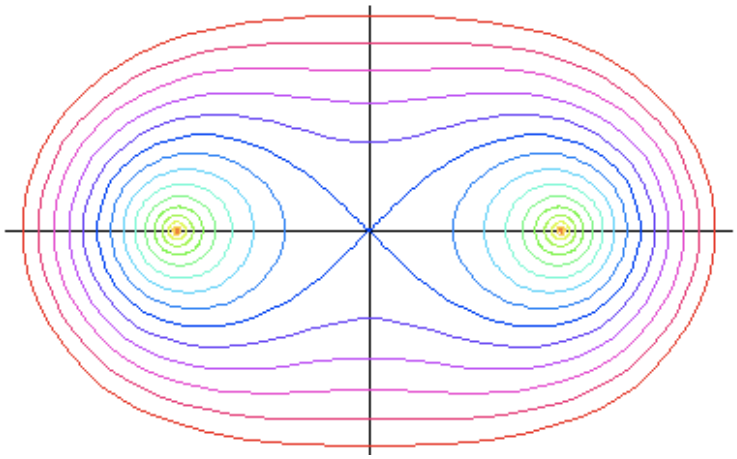
First, more remarks on the Lemniscate: A *Cassini Oval*[1] is the locus of points z whose distance from two given points a and b is a constant. If we take $b = -a$ on the real axis, then the equation of a Cassini oval is

$$|z^2 - a^2| = \text{constant}.$$

For various values of the constant, the Cassini ovals look like [2]

[1] 1625-1712, Italian Astronomer, name of NASA probe currently nearing Saturn.

[2] picture stolen from MathWorld.



The Lemniscate, the figure-eight in the middle, is the Cassini oval when the constant equals a^2. In this problem, we have $a^2 = 1/2$.

(a) Use the parametrization:

$$x(t) = \sqrt{\frac{t^2+t^4}{2}}, \qquad y(t) = \sqrt{\frac{t^2-t^4}{2}},$$

to show that the arclength of the lemniscate is 2ϖ.

(b) Express ϖ in terms of π and $\Gamma(1/4)$ (see previous exercise).

(c) Let $a_1 = \sqrt{2}, b_1 = 1$. Define (a_n, b_n) recursively by

$$a_n = (a_{n-1} + b_{n-1})/2, \qquad b_n = (a_{n-1} \cdot b_{n-1})^{1/2}.$$

compute (a_6, b_6) and π/ϖ to several decimal places.

As t varies from 0 to 1, the curve $\gamma(t) = x(t) + iy(t)$ traces out the part of the Lemniscate in the quadrant $x > 0, y > 0$, which is one fourth of the total arclength. We have

$$x'(t)^2 + y'(t)^2 = \frac{1}{1-t^4},$$

so the total arclength is

$$4\int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt = 4\int_0^1 \frac{dt}{1-t^4} = 4 \cdot \frac{\varpi}{2} = 2\varpi.$$

From the calculation of I_0 in the previous problem, we have

$$\varpi = 2I_0 = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} = 2.62205755429211981046484 0\ldots,$$

so

$$\frac{\pi}{\varpi} = 1.198140234735592207439922\ldots.$$

Meanwhile the Arithmetic-Geometric Mean between 1 and $\sqrt{2}$ gives

n	a_n	b_n
1	1.414213562373095048801689	1.000000000000000000000000
2	1.207106781186547524400844	1.189207115002721066717500
3	1.198156948094634295559172	1.198123521493120122606586
4	1.198140234793877209082879	1.198140234677307205798384
5	1.198140234735592207440631	1.198140234735592207439214
6	1.198140234735592207439922	1.198140234735592207439922