

MT815 Complex Variables Homework III

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Exercise 1. Verify the functional equation

$$\frac{\zeta(1-s)}{\zeta(s)} = 2 \cdot (2\pi)^{-s} \cdot \Gamma(s) \cdot \cos(s\pi/2)$$

for $s = 1 - 2k$ with k a positive integer.

From Euler's evaluations of $\zeta(1 - 2k)$ and $\zeta(2k)$, we have, for $s = 1 - 2k$:

$$\frac{\zeta(1-s)}{\zeta(s)} = \frac{\zeta(2k)}{\zeta(1-2k)} = \frac{(2\pi)^{2k} B_k}{2 \cdot (2k)!} \cdot (-1)^k \frac{2k}{B_k} = \frac{(-1)^k (2\pi)^{2k}}{2 \cdot (2k-1)!}$$

on the other hand,

$$\begin{aligned} \lim_{t \rightarrow k} 2 \cdot (2\pi)^{2t-1} \Gamma(1-2t) \cos((1-2t)\pi/2) \\ &= 2 \cdot (2\pi)^{2k-1} \lim_{t \rightarrow k} \frac{\pi \sin(\pi t)}{\sin(2\pi t) \Gamma(2t)} \\ &= \frac{(2\pi)^{2k}}{2 \cos(k\pi) \Gamma(2k)} \\ &= \frac{(-1)^k (2\pi)^{2k}}{2 \cdot (2k-1)!}. \end{aligned}$$

Exercise 2. Compute

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^s}$$

in terms of $\zeta(s)$, for $\Re(s) > 1$.

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2k)^s} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} \\ &= 2^{-s}\zeta(s) + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s}.\end{aligned}$$

So

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} = (1 - 2^{-s})\zeta(s).$$

Exercise 3. Use Exercise 2 to prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

converges to $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) > 0$ (not just $\Re(s) > 1$) and that the function $(1 - 2^{1-s})\zeta(s)$ is entire.

For $\Re(s) > 1$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} - \sum_{k=1}^{\infty} \frac{1}{(2k)^s} \\ &= (1 - 2^{-s})\zeta(s) - 2^{-s}\zeta(s) \\ &= (1 - 2^{1-s})\zeta(s).\end{aligned}$$

The factor $1 - 2^{1-s}$ cancels the pole of $\zeta(s)$ at $s = 1$, so the function $(1 - 2^{1-s})\zeta(s)$ is entire, and in particular is analytic on $\Re(s) > 0$. It therefore suffices to show

that the series converges uniformly on compact subsets of $\Re(s) > 0$. Let $\epsilon > 0$ and let K be a compact subset of $\Re(s) > 0$. For $z = 2x + iy \in K$, let

$$a_n = \frac{(-1)^n}{n^x}, \quad b_n = \frac{1}{n^{x+iy}}.$$

We must show that the sum $S_{p,q}$ defined by

$$S_{p,q} = \sum_{n=p}^q a_n b_n$$

is $< \epsilon$ for $p < q$ sufficiently large. We use summation by parts:

$$S_{p,q} = \sum_{n=p}^{q-1} A_{p,n}(b_n - b_{n+1}) + A_{p,q}b_q,$$

where

$$A_{p,n} = \sum_{k=p}^n a_k.$$

The series $\sum a_n$ converges uniformly for $z \in K$ by the alternating test. Hence there is $N > 0$ such that

$$A_{p,n} < \epsilon$$

for all $p \geq N$ and all $z \in K$. Now

$$b_n - b_{n+1} = \frac{1}{n^{x+iy}} - \frac{1}{(n+1)^{x+iy}} = (x+iy) \int_{\log(n+1)}^{\log n} e^{-t(x+iy)} dt,$$

so

$$|b_n - b_{n+1}| \leq |x+iy| \int_{\log(n+1)}^{\log n} e^{-tx} dt = \left| \frac{x+iy}{x} \right| \left(\frac{1}{n^x} - \frac{1}{(n+1)^x} \right).$$

This will make the sum $S_{p,q}$ telescope down to one term. Note also that

$$\left| \frac{x+iy}{x} \right| = \sec \theta,$$

where $\theta = \arg(x+iy) \in (-\pi/2, \pi/2)$. Since $|\theta|$ is bounded away from $\pi/2$ for $z \in K$, there is a constant M so that

$$\left| \frac{x+iy}{x} \right| \leq M$$

for $z \in K$. Hence for $p \geq N$ we have

$$|S_{p,q}| \leq \epsilon \left(M \sum_{n=p}^{q-1} \left(\frac{1}{n^x} - \frac{1}{(n+1)^x} \right) + \frac{1}{q^x} \right) = \frac{\epsilon M}{p^x}.$$

Increasing N if necessary, we have $N^x < M$ for all $z \in K$. Then for $p \geq N$ we have $|S_{p,q}| \leq \epsilon$, as desired.

Exercise 4. Show that the function

$$\xi(s) = (s-1)\Gamma\left(1 + \frac{s}{2}\right) \pi^{-s/2} \zeta(s)$$

has the following properties:

- (a) $\xi(s)$ is entire.
- (b) The zeros of $\xi(s)$ are the zeros of $\zeta(s)$ lying in the strip $0 \leq \Re(s) \leq 1$.
- (c) $\xi(1-s) = \xi(s)$.
- (d) $\xi(0) = \frac{1}{2}$.

The factor $(s-1)$ cancels the pole at $s=1$ in $\zeta(s)$. The remaining possible poles of $\xi(s)$ come from the poles of $\Gamma(1 + \frac{s}{2})$, which occur at negative even integers $s = -2n$, and these are simple poles. Since $\zeta(-2n) = 0$, these poles cancel as well. Hence $\xi(s)$ is entire.

Since the zeros of $\zeta(s)$ at the points $s = -1 - 2n$ are simple, the product $\Gamma(1 + \frac{s}{2})\zeta(s)$ is nonzero at $s = -2n$. Likewise $(s-1)\zeta(s) = 1$ at $s = 1$. So the zeros of $\xi(s)$ are the remaining zeros of $\zeta(s)$, and these lie in the strip $0 \leq \Re(s) \leq 1$. Part (d) follows from the fact that $\zeta(0) = -1/2$.

For the functional equation, we use the functional equation for $\zeta(s)$ to compute

$$\begin{aligned} \xi(1-s) &= (-s)\Gamma\left(1 + \frac{1-s}{2}\right) \pi^{(s-1)/2} \zeta(1-s) \\ &= -s \cdot \frac{1-s}{2} \Gamma\left(\frac{1-s}{2}\right) \pi^{(s-1)/2} \cdot 2^{1-s} \pi^{-s} \Gamma(s) \cos\left(\frac{s\pi}{2}\right) \zeta(s) \\ &= s(s-1)2^{-s} \pi^{(-s-1)/2} \Gamma\left(\frac{1-s}{2}\right) \Gamma(s) \cos\left(\frac{s\pi}{2}\right) \zeta(s). \end{aligned}$$

We must show that this is equal to

$$\xi(s) = (s-1)\Gamma\left(1 + \frac{s}{2}\right) \pi^{-s/2} \zeta(s) = (s-1) \frac{s}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s)$$

Hence we must show that

$$2^{1-s} \pi^{-1/2} \Gamma\left(\frac{1-s}{2}\right) \Gamma(s) \cos\left(\frac{s\pi}{2}\right) = \Gamma\left(\frac{s}{2}\right) \quad (1)$$

From the duplication formula we have

$$\Gamma(s) = (2\pi)^{-1/2} 2^{s-1/2} \Gamma\left(\frac{1+s}{2}\right) \Gamma\left(\frac{s}{2}\right).$$

The left side of (1) becomes

$$2^{1-s} \pi^{-1/2} \Gamma\left(\frac{1-s}{2}\right) (2\pi)^{-1/2} 2^{s-1/2} \Gamma\left(\frac{1+s}{2}\right) \Gamma\left(\frac{s}{2}\right) \cos\left(\frac{s\pi}{2}\right),$$

which simplifies to

$$\pi^{-1} \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) \Gamma\left(\frac{s}{2}\right) \cos\left(\frac{s\pi}{2}\right). \quad (2)$$

From the identity $\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z$, we have

$$\Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) = \frac{\pi}{\sin(\pi(1-s)/2)} = \frac{\pi}{\cos(s\pi/2)}.$$

Hence the expression in (2) equals $\Gamma(s/2)$, as we wished to show.

Exercise 5. A subset $S \subset \mathbf{N}$ has *analytic density*

$$\delta(S) = \lim_{s \rightarrow 1} \frac{\sum_{n \in S} n^{-s}}{\sum_{n \in \mathbf{N}} n^{-s}},$$

provided the limit exists. When it exists, we have $0 \leq \delta(S) \leq 1$, and $\delta(S)$ represents, in some sense, the probability that a randomly chosen positive integer lies in S .

Let \square be the set of square-free numbers, i.e., \square is the set of positive integers n of the form $n = p_1 p_2 \cdots p_k$ where the p_i are distinct primes. Show that

$$\sum_{n \in \square} n^{-s} = \prod_p (1 + p^{-s}). \quad (3)$$

Then use this formula to compute $\delta(\square)$ as a value of $\zeta(s)$ and thus compute the probability that a randomly chosen positive integer is square-free.

Equation (3) follows from the fact that every square-free number is uniquely expressed as a product of distinct primes. Then using the analogous product formula for $\zeta(s)$, we have

$$\frac{\sum_{n \in \square} n^{-s}}{\sum_{n \in \mathbf{N}} n^{-s}} = \frac{\prod_p (1 + p^{-s})}{\prod_p (1 - p^{-s})^{-1}} = \prod_p (1 - p^{-2s}).$$

As $s \rightarrow 1$, this converges to

$$\prod_p (1 - p^{-2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} = \frac{607927}{1,000,000}$$

According to Mathematica, the number of square-free positive integers between 1 and 1,000,000 is actually 607926.