MT815 Complex Variables Homework IV

Witch 2

Due Friday April 4

Exercise 1. Let L be a lattice in $\mathbf{C}$, let

$$G_k(L) = \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^k}$$

be the corresponding Eisenstein series and let

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in L} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right] = \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1) G_{n+2}(L) z^n$$

be the Weierstrass $\wp$-function for L, which satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

where $g_2 = 60G_4(L)$ and $g_3 = 140G_6(L)$.

(a) Prove that if $iL = L$ then $G_k(L) = 0$ unless 4 divides k.

(b) Show that $\wp(iz) = -\wp(z)$ if $iL = L$.

(c) Suppose now that $\rho L = L$, where $\rho = e^{2\pi i/3}$. State and prove results analogous to (a) and (b).

For any $\alpha \in \mathbf{C}^\times$ we have $G_k(\alpha L) = \alpha^{-k} G_k(L)$. So if $\alpha L = L$ we have $G_k(L) = 0$ unless $\alpha^k = 1$.

If $iL = L$ then

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} (4m-1) G_{4m} z^{4m-2}.$$

Since each power of z is $\equiv 2 \mod 4$, we have $\wp(iz) = i^2\wp(z) = -\wp(z)$.

If $\rho L = L$ then

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty}(6m-1)G_{6m}z^{6m-2}.$$

Since each power of z is $\equiv 2 \mod 6$, we have $\wp(\rho z) = \rho^{-2}\wp(z)$.

Exercise 2. Prove that if L is any lattice in $\mathbf{C}$ closed under complex conjugation, and $x \in \mathbf{R}$ is not in L, then $\wp_L(x) \in \mathbf{R}$.

If $\bar{L} = L$ then $\overline{G_k(L)} = G_k(\bar{L}) = G_k(L)$, so each $G_k(L)$ is real. Hence the Laurent series for $\wp(z)$ at $z = 0$ has real coefficients. This implies that $\wp(x) \in \mathbf{R}$ for $x \in \mathbf{R} - L$.

Exercise 3. Let $L = \mu\mathbf{Z}[i]$, where $\mathbf{Z}[i] = \{n + mi : \ n, m \in \mathbf{Z}\}$ is the lattice of Gaussian integers and μ is any positive real number. Find the zeros of $\wp = \wp_L$ and then prove that $g_2(L) > 0$.

The differential equation for $\wp$ is

$$(\wp')^2 = 4\wp^3 - g_2(L)\wp.$$

So the zeros of $\wp$ are among the zeros of $\wp'$, which we know are

$$z_1 = \mu/2, \qquad z_2 = \mu i/2, \qquad z_3 = \mu(1+i)/2.$$

Since $iz_3 \equiv z_3 \mod L$, and $\wp(iz) = -i\wp(z)$, we have $\wp(z_3) = 0$. If $\wp(z_1) = 0$, then also $\wp(z_2) = \wp(iz_1) = 0$, which is too many zeros for $\wp$. Hence $\wp$ has a double zero at z_3, and $\wp(z_1) \neq 0$. Since $\bar{L} = L$, we know that $\wp(0, \mu) \subset \mathbf{R}$, so $\wp(z_1) \in \mathbf{R}$, hence $\wp(z_2) = -\wp(z_1) \in \mathbf{R}$. and also $g_2(L) \in \mathbf{R}$. Since the polynomial $4x^3 - g_2(L)x$ has the three distinct roots $\wp(z_1), \wp(z_2), \wp(z_3)$, all of which are real, we must have $g_2(L) > 0$.

Exercise 4. Let f be an elliptic function with period lattice L. View f as a function $f : \mathbf{C}/L \to \mathbf{C}$. The *order* of f is the number m of poles of f in $\mathbf{C}/L$, counting multiplicities. That is, m is the sum of the orders of the poles of f in $\mathbf{C}/L$, or if you prefer, in any shifted period parallelogram whose sides miss all poles and zeros of f.

(a) Prove that for all $w \in \mathbf{C}$, the function f takes the value w exactly m times in $\mathbf{C}/L$, counting multiplicities.

(b) What is the order of $\wp = \wp_L$? Given $z \in \mathbf{C} - L$, find z^* in $\mathbf{C} - L$ such that

$$\wp(z^*) = \wp(z) \quad \text{and} \quad \wp'(z^*) = -\wp'(z).$$

(c) Use (a) and (b) to prove that $(\wp, \wp')$ maps $\mathbf{C} - L$ *onto* the curve

$$C = \{(u, v) \in \mathbf{C}^2 \mid v^2 = 4u^3 - g_2 u - g_3\}.$$

The elliptic function $g(z) - w$ has the same poles as f, hence also has order m, so $g(z)$ has m zeros. The simplest value for z^* is $z^* = -z$. If $z = t\lambda_1 + s\lambda_2$ is inside a period parallelogram Π spanned by λ_1 and λ_2, and we also want to have $z^* \in \Pi$, then we must take

$$z^* = (1 - t)\lambda_1 + (1 - s)\lambda_2.$$

Now let (u, v) be a point on the curve C. By part (a) there exists $z \in \mathbf{C} - L$ such that $\wp(z) = u$. Since

$$\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3 = 4u^3 - g_2 u - g_3 = v^2,$$

we have $\wp'(z) = \pm v$. Replacing z by z^* if necessary, we can arrange $\wp(z) = v$.

Exercise 5. Using exercise 4, show that there is a unique $\mu > 0$ so that $g_2(L) = 4$, so that $\wp = \wp_L$ has differential equation $(\wp')^2 = 4(\wp^3 - \wp)$. Then show that $\wp'' = 6\wp^2 - 2$ and use this to draw (or describe) the graph of $\wp(x)$ for $x \in (0, \mu)$.

We have

$$g_2(L) = \mu^{-4} g_2(\mathbf{Z}[i]) > 0,$$

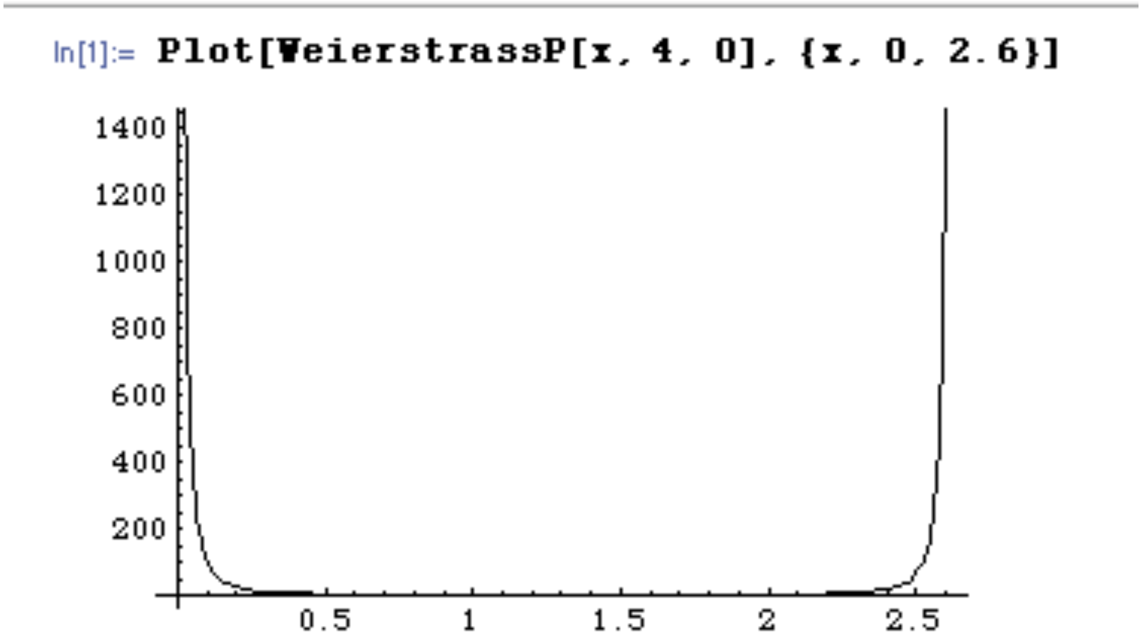
so $g_2(\mathbf{Z}[i]) > 0$ and there is a unique $\mu > 0$ such that

$$4\mu^4 = g_2(\mathbf{Z}[i]),$$

making $g_2(L) = 4$ and we have the equation $(\wp')^2 = 4(\wp^3 - \wp)$. Differentiating both sides gives $\wp'' = 6\wp^2 - 2$. At $\mu/2$, we have

$$\wp(\mu/2) = 1, \quad \wp'(\mu/2) = 0, \quad \wp''(\mu/2) = 4 > 0,$$

and there are no other zeros of $\wp'$ in the interval $(0, \mu)$. It follows that $\wp$ is concave up in $(0, \mu)$, with a minimum at $\mu/2$.



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In[1]:= Plot[WeierstrassP[x, 4, 0], {x, 0, 2.6}]
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Exercise 6. Let μ be as in the previous exercise. For $x \in [0, \mu/2]$, consider the function

$$F(x) = \int_{\wp(x)}^{\infty} \frac{du}{\sqrt{4u^3 - 4u}},$$

where the integral is over the real interval $[\wp(x), \infty)$ and the square-root is chosen to be positive, except at $u = 1$. (Why can we do this?)

(a) Show that $F(x) = x$. (Use the Fundamental Theorem of Calculus!)

(b) Compute $\int_1^\infty (u^3 - u)^{-1/2} du$. (Let $t = 1/u^2$.)

(c) Compute μ.

(d) Compute $G_4(\mathbf{Z}[i])$.

Since $\wp(0, \mu/2) = (1, \mathbf{R})$, on which the function $u^3 - u \geq 0$, we can choose the postive real square root. Then

$$F'(x)^2 = (\wp'(x)^2 \cdot (4\wp(x)^3 - 4\wp(x))^{-1} = 1,$$

so $F'(x) = \pm 1$ is a constant. Since $\wp(x)$ is decreasing on $(0, \mu/2)$, the sign is $+$ and since $\wp(0) = \infty$ we have $F(0) = 0$, so $F(x) = x$. Setting $x = \mu/2$, we have

$$\begin{aligned}\mu &= 2\int_1^\infty \frac{du}{\sqrt{4u^3 - 4u}} = \int_1^\infty \frac{du}{\sqrt{u^3 - u}} \\ &= \frac{1}{2}\int_0^1 t^{-3/4}(1-t)^{-1/2}dt \\ &= \frac{\Gamma(1/4)\Gamma(1/2)}{2 \cdot \Gamma(3/4)} = \frac{\Gamma(1/4)^2}{2\sqrt{2\pi}} \\ &= \varpi.\end{aligned}$$

Alternatively, we can turn the integral for μ directly into the Lemniscatic integral by the substitution $u = v^{-2}$, which gives

$$\int_1^\infty \frac{du}{\sqrt{u^3 - u}} = 2\int_0^1 \frac{dv}{\sqrt{1 - v^4}} = 2 \cdot \frac{\varpi}{2} = \varpi.$$

From the equation $g_2(\mu\mathbf{Z}[i]) = 4$, we have

$$\mu^{-4} 60 G_4(\mathbf{Z}[i]) = 4,$$

or

$$G_4(\mathbf{Z}[i]) = \frac{\varpi^4}{15}.$$

COMMENTS: It's a Swiss Thing

Recall Euler's result from the mid 1700's that

$$\sum_{n=1}^\infty \frac{1}{n^{2k}} = \frac{(2\pi)^{2k} B_k}{2 \cdot (2k)!}$$

which we could write as

$$\sum_{0 \neq n \in \mathbf{Z}} \frac{1}{n^{2k}} = \frac{(2\pi)^{2k} B_k}{(2k)!}, \tag{1}$$

where

$$\pi = 2\int_0^1 \frac{dt}{\sqrt{1 - t^2}}$$

and the Bernoulli numbers B_k are defined as the coefficients in the expansion of the meromorphic function

$$\cot z = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{2^{2k} B_k}{2k} \frac{z^{2k-1}}{(2k-1)!} \tag{2}$$

whose period group is $\pi\mathbf{Z}$.

In 1897 the Swiss mathematician Adolf Hurwitz (following his Swiss predecessors Euler and the Bernoullis) found an analogous formula for the sum

$$G_{4k}(\mathbf{Z}[i]) = \sum_{0 \neq \lambda \in \mathbf{Z}[i]} \frac{1}{\lambda^{4k}},$$

with the cotangent function replaced by the Weierstrass function $\wp$ of that lattice L for which

$$(\wp')^2 = 4\wp^3 - 4\wp. \tag{3}$$

In the exercise above, you found that

$$L = \varpi \mathbf{Z}[i],$$

where

$$\varpi = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}}.$$

Proceeding analogously to the expansion (2), Hurwitz defines numbers that we will call H_k, by the expansion

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{2^{4k} H_k}{4k} \frac{z^{4k-2}}{(4k-2)!}.$$

Substituting this into the equation (3), Hurwitz found the recursion formula

$$H_k = \frac{3}{(2k-3)(16k^2-1)} \sum_{j=1}^{k-1} (4j-1)(4k-4j-1)\binom{4k}{4j} H_j H_{k-j}$$

whose first few values are

$$H_1 = \frac{1}{10}, \qquad H_2 = \frac{3}{10}, \qquad H_3 = \frac{3^4 \cdot 7}{10 \cdot 13}.$$

Since

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(4k-1)G_{4k}(\varpi\mathbf{Z}[i])z^{4k-2},$$

It follows that

$$G_{4k}(\mathbf{Z}[i]) = \frac{(2\varpi)^{4k}}{(4k)!}H_k,$$

in analogy with Euler's result (1).

We have not discussed the number theory of Bernoulli numbers, in particular the von Staudt-Clausen Theorem on the denominators of B_k (see Ireland-Rosen *A Classical Introduction to Modern Number Theory).* Hurwitz found analogues of these results as well. His paper (in German, but readable by us thanks to consistency of notation over the years) is on the course website.