MT830 Representation Theory Homework I

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Exercise 1.1 In this exercise, you will complete the calculation of the character table of the symmetric group $\mathfrak{S}_3$.

The conjugacy classes in $\mathfrak{S}_3$ are:

$$[111] = \{e\}, \quad [21] = \{(12), (23), (13)\}, \quad [3] = \{(123), (321)\}.$$

The irreducible characters are
χ_3 = the trivial character,
χ_{21} = the reflection character,
χ_{111} = the sign character.

The character table is given by

	[111]	[12]	[3]
χ_3:	1	1	1
χ_{21}:	2	0	-1
χ_{111}:	1	-1	1

Note that any two irreducible characters χ, χ' satisfy the orthogonality relations

$$\frac{1}{|\mathfrak{S}_3|} \sum_{s \in \mathfrak{S}_3} \chi(s)\chi'(s) = \begin{cases} 1 & \text{if } \chi = \chi' \\ 0 & \text{if } \chi \neq \chi'. \end{cases}$$

Exercise 1.1a Verify the entries for χ_{21} above.

Exercise 1.1b Verify the orthogonality relations above.

Exercise 1.2 For $n \geq 2$, let $V_n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \sum_{i=1}^n x_i = 0\}$. Then V_n is an invariant subspace of the permutation representation of $\mathfrak{S}_n$ on $\mathbb{C}^n$, giving a representation $\rho_n : \mathfrak{S}_n \to GL(V_n)$, called the *reflection representation* of $\mathfrak{S}_n$.

a) Show that each transposition (i, j) has exactly one eigenvalue on V_n equal to -1, and $n-2$ eigenvalues equal to $+1$.

Hint: all transpositions are conjugate. Also it may be easier to find the eigenvalues on the permutation representation first.

b) The vector $\alpha_i = e_i - e_{i+1}$ is a -1 eigenvector for the transposition $(i, i+1)$. Prove that if U is an invariant subspace of V_n then U contains at least one of these vectors α_i.

c) Prove that if U is an invariant subspace of V_n then U contains all of these vectors α_i. Conclude that $U = V_n$, so V_n is irreducible.

d) Compute the character $\operatorname{tr} \rho_n(s)$ in terms of the cycle type of $s \in \mathfrak{S}_n$.

Hint: Compute the character of the permutation representation first.

e) Let $\epsilon : \mathfrak{S}_n \to \{\pm 1\}$ be the sign character of $\mathfrak{S}_n$ and let $\epsilon\rho_n : \mathfrak{S}_n \to GL_(V_n)$ be the representation on the same space V_n but with the new action given by $\epsilon\rho_n(s) = \epsilon(s)\rho_n(s)$. Prove that $\epsilon\rho_n$ is equivalent to ρ_n iff $n = 2$.

Hint: Compare the values of $\rho_n(s)$ and $\epsilon\rho_n(s)$ for a transposition $s \in \mathfrak{S}_n$.

Exercise 1.3 In this exercise, take $G = \mathfrak{S}_4$ and assume the basic results that $|\operatorname{Irr}(G)|$ equals the number of conjugacy classes of G and that

$$\sum_{\rho \in \operatorname{Irr}(G)} \dim \rho^2 = |G|.$$

a) Show that $\operatorname{Irr}(G) = \{1_G, \epsilon, \rho_4, \epsilon\rho_4, \psi\}$ where all but ψ are irreducible representations you have already discovered and ψ, whatever it is, has dimension two.

b) Let C be the conjugacy-class of 22-cycles in G. Show that the action of G on C by conjugation gives a surjective homomorphism $\mathfrak{S}_4 \to \mathfrak{S}_3$ and use this to find ψ.

c) Compute the character table of $\mathfrak{S}_4$. Label the conjugacy classes as we did for

$\mathfrak{S}_3$ above, and label the irreducible characters by:

$$\begin{aligned}
\chi_4 &= 1_G \\
\chi_{31} &= \operatorname{tr} \rho_4 \\
\chi_{22} &= \operatorname{tr} \psi \\
\chi_{211} &= \operatorname{tr} \epsilon\rho_4 \\
\chi_{1111} &= \epsilon.
\end{aligned}$$

(This labelling will be explained later.) With these labellings, the character table of $\mathfrak{S}_4$ looks like

	1111	211	22	31	4
χ_4					
χ_{31}					
χ_{22}					
χ_{211}					
χ_{1111}					

and you must fill in the squares.

Exercise 1.4 Let $G = D_8$ be the dihedral group of order eight. Compute the conjugacy classes and character table of G.