MT830 Representation Theory Homework II

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Exercise 2.1 Prove that all irreducible complex representations of a finite abelian group are one dimensional. (Suggestion: Use Schur's Lemma.)

Exercise 2.2 Let $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ be two representations of a finite group G on finite dimensional vector spaces V, V'. We get two new representations

$$\rho^* \otimes \rho' : G \to GL(V^* \otimes V'), \qquad \rho^* \otimes \rho'(g) = \rho^*(g) \otimes \rho'(g),$$

and

$$R : G \to GL(\mathrm{Hom}(V, V')), \qquad R(g)T = \rho'(g)T\rho(g)^{-1}.$$

Prove the the linear isomorphism

$$\mathcal{T} : V^* \otimes V' \to \mathrm{Hom}(V, V'), \qquad \mathcal{T}(\lambda \otimes v')(v) = \lambda(v)v'$$

is an equivalence of representations of G.

Exercise 2.3a Let the finite group G act on a finite set X. Prove the Burnside Counting Lemma:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

without character theory, by counting the set $\{(g, x) :\ g \in G,\ x \in X,\ g \cdot x = x\}$ in two ways.

Exercise 2.3b For each $\sigma \in \mathfrak{S}_5$, find the cycle type of the permutation σ_2 induced by σ on the set Λ^2_5 of two-element subsets of $\{1, 2, 3, 4, 5\}$ and use this to compute the number of graphs on five vertices. Check the result, either by drawing all the graphs, or using Mathematica.

Exercise 2.4a In the last homework, you found all the irreducible characters of $\mathfrak{S}_4$. Which of these is the character of the representation of $\mathfrak{S}_4$ as rotational symmetries of the cube in $\mathbb{R}^3$?

Exercise 2.4b The group $\mathfrak{S}_4$ acts on the set Λ_4^2 of two-element subsets of $\{1, 2, 3, 4\}$. Decompose the character of the permutation representation on Λ_4^2 as a sum of irreducible characters of $\mathfrak{S}_4$. Do the same with the permutation representation of $\mathfrak{S}_4$ on Λ_4^3.

Exercise 2.5 Let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$ be two partitions of n, where we make some parts equal to zero, in order to always have n parts. (For example, write 31 as 3100.) Define $\lambda \leq \mu$ if

$$\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$$

for all $1 \leq k \leq n$. Write out the ordering of the partitions of $n = 4$ and use the decompositions on the permutation representations of $\mathfrak{S}_4$ on Λ_4^k to explain the labelling

$$\begin{aligned} \chi_4 &= 1_G \\ \chi_{31} &= \operatorname{tr} \rho_4 \\ \chi_{22} &= \operatorname{tr} \psi \\ \chi_{211} &= \operatorname{tr} \epsilon\rho_4 \\ \chi_{1111} &= \epsilon \end{aligned}$$

from the previous homework.