MT830 Representation Theory Homework III

October 9, 2008

Exercise 3.1 Let χ be the character of an irreducible representation $\rho : G \to GL(V)$.

a) Prove that for all $g \in G$ we have $|\chi(g)| \leq \chi(1)$, with equality if and only if $\rho(g) = zI$ is a scalar matrix.

Hint: For $z_1, \ldots, z_n$ in $\mathbb{C}$ we have $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$, with equality iff each $z_k = t_k z_1$ for some positive real number t_k.

b) Show that $\{g \in G : \ |\chi(g)| = \chi(1)\}$ is a normal subgroup of G.

Hint: Consider the composition $G \xrightarrow{\rho} GL(V) \to PGL(V)$, where $PGL(V)$ is the quotient of $GL(V)$ by the subgroup of scalar transformations.

c) Prove that if $\chi(g) = \chi(1)$ for all irreducible characters χ of G then $g = 1$.

Hint: Consider the regular representation of G.

Exercise 3.2 Prove that the number of one-dimensional representations of G equals the index $[G : G']$ of the commutator subgroup G' of G.

Exercise 3.3 Let H be a normal subgroup of G. Then each irreducible representation $\rho' : G/H \to GL(V)$ gives an irreducible representation of G, via the composition $\rho : G \to G/H \xrightarrow{\rho'} GL(V)$. The representations ρ obtained in this way are exactly those with $H \subset \ker \rho$. Now let $x \in G$ be an element whose centralizer $C_G(x)$ has the property that $H \cap C_G(x) = \{1\}$. Prove that if $H \not\subset \ker \rho$ then $\chi_\rho(x) = 0$. Also illustrate this with some examples.

Hints: First show that $C_G(x)$ is isomorphic to a subgroup of $C_{G/H}(xH)$. Next, break the sum in the second orthogonality relation for G into two parts: one with $H \subset \ker \rho$ and one with $H \not\subset \ker \rho$, then use the second orthogonality relation for $C_{G/H}(xH)$.