

MT830 Homework IV

due Wednesday November 5

Exercise 4.1 Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order eight. Prove that there is a unique automorphism $\sigma \in \text{Aut}(Q)$ such that

$$\sigma(i) = j, \quad \sigma(j) = k, \quad \sigma(k) = i.$$

In what follows, let

$$\tilde{A}_4 = Q \rtimes \langle \sigma \rangle.$$

This is a nonabelian group of order 24, called the *binary tetrahedral group*. In how many ways can you see that \tilde{A}_4 is not isomorphic to the symmetric group S_4 ?

Exercise 4.2 Prove that $\tilde{A}_4/\{\pm 1\} \simeq A_4$.

Exercise 4.3 Let G be a group with the following properties:

1. $|G| = 24$
2. The 2-Sylow subgroup of G is isomorphic to Q and is normal in G .
3. The 3-Sylow subgroup of G is not normal in G .

Prove that $G \simeq \tilde{A}_4$.

Exercise 4.4 Let G be the subgroup of S^3 whose elements are those of Q together with those of the form $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$, with all possible independent choices of signs. Prove that $G \simeq \tilde{A}_4$.

Exercise 4.5 Let $SL_2(3)$ be the group of matrices with entries in the finite field \mathbb{F}_3 and determinant equal to one. Prove that $SL_2(3) \simeq \tilde{A}_4$. Use the action of $SL_2(3)$ on the set of lines in \mathbb{F}_3^2 to give another proof of the isomorphism $\tilde{A}_4/\{\pm 1\} \simeq A_4$.

Exercise 4.6 Give the character table of \tilde{A}_4 .