

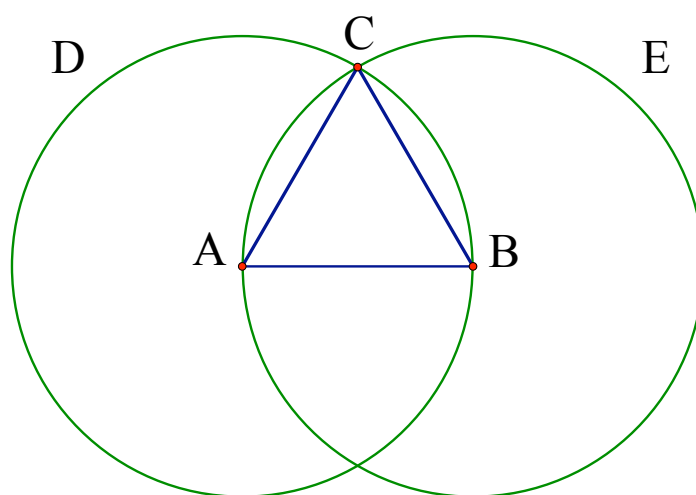
**MT 453 Elements Day 1**

**Speaker: Alice**

**Scribes: Bob, Cleopatra**

**Proposition I.1**

*How to construct an equilateral triangle whose base is a given line segment.*



Let  $A$  and  $B$  be the ends of the segment.

Draw the circle  $D$  with center  $A$  and radius  $AB$ . (post. 3)

Draw the circle  $E$  with center  $B$  and radius  $BA$ . (post. 3)

Let  $C$  be a point where the circles meet.

Then this point  $C$  is on both circles  $D$  and  $E$ .

Since  $C$  is on the circle  $D$ , we have  $AC = AB$ . (def. 15)

Since  $C$  is on the circle  $E$ , we have  $BC = AB$ . (def. 15)

But things equal to the same thing are equal to each other. (c.n. 1)

Hence  $AB = AC = BC$ , so the triangle  $ABC$  is equilateral. (def. 20)

Q.E.F.

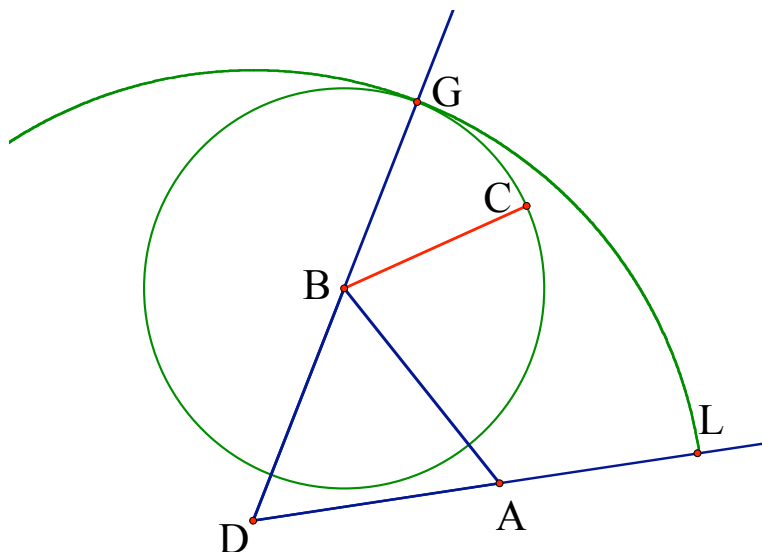
**Comments:** 1. Equilateral triangles are not part of the five axioms, so their existence must be shown to be a consequence of the axioms. This is done by actually constructing equilateral triangles. While this may be a fun activity, the existence is the main point.

2. When Euclid says that the circles meet, he is relying on some common sense which is not part of his axioms. He implicitly asserts that if a point  $P$  starts at  $B$  and moves along the circle  $D$ , then the distance  $BP$ , which starts at zero and increases beyond  $AB$ , must at some point equal  $AB$ . Nowadays, we call this the Intermediate Value Theorem, which is based on the Completeness Property of the real numbers.

3. The circles actually meet in two points. Euclid's construction produces both equilateral triangles with base  $AB$ .

### Proposition I.2

*How to construct a line segment of a given length at a given point.*



Let  $A$  be the given point and let  $BC$  be a line segment of the given length.  
Using compass and straightedge, we have to construct a line segment of length  $BC$  with one end of the segment at  $A$ .  
Draw the line segment  $AB$ . (post. 1)  
Construct an equilateral triangle  $ABD$  on  $AB$ . (I.1)  
Draw the circle with center  $B$  and radius  $BC$ . (post. 3)  
Extend the line  $DB$  through  $B$  until it cuts the circle just drawn, at the point  $G$ . (post. 2)  
Draw the circle with center  $D$  and radius  $DG$ . (post. 3)  
Extend the line  $DA$  through  $A$  until it cuts the circle just drawn, at the point  $L$ . (post. 2)  
Then  $BC = BG$  since both are radii of the circle with center  $B$ .  
Likewise,  $DL = DG$  since both are radii of the circle with center  $D$ . (def. 15)  
Subtracting the equals  $DA = DB$  from  $DL = DG$ , we have  $AL = BG$ . (c.n.3)  
Since  $BC = BG$  and  $AL = BG$ , we have  $BC = AL$ . (c.n. 1)  
Therefore,  $AL$  is a line of the required length. Q.E.F.

### Comments

1. The constructed segment  $AL$  is not parallel to the given  $BC$ . What is the angle between these two segments?
2. In this picture  $DB$  already meets the circle with center  $B$ . But it will not always meet this circle, since  $DB = DA$ , which could be very short. But the other intersection, at  $G$ , always exists (again, using some kind of intermediate value notion).