MT 453 Elements Day 8

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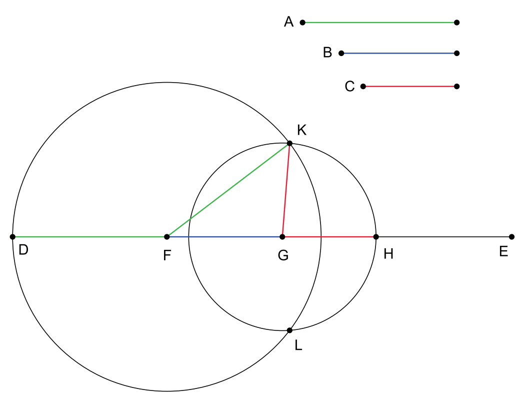
Proposition I.22

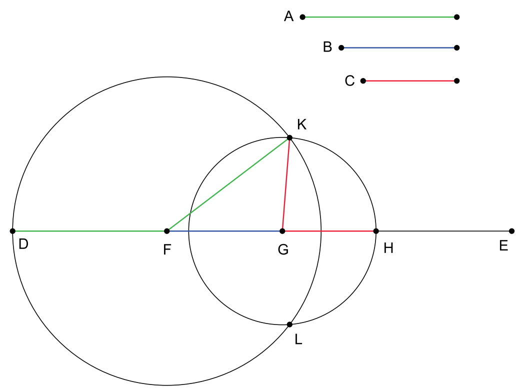
To construct a triangle with the sides equal to three given lines, provided that the sum of any two of the lines is greater than the third line [I.20].



Let A, B, and C be the given lines.
Draw line DE with endpoint D and extending indefinitely towards E. [Post.2]
Cut DE at F so that $DF = A$, cut FE at G so that $FG = B$, and cut GE at H so that $GH = C$. [Prop.I.3]
Draw circle DKL with center F and radius DF, and draw circle HKL with center G and radius GH. [Post.3]
Draw straight lines FK and GK. [Post.1]
Claim: $\triangle FGK$ has sides equal to A, B, and C.

$FK = FD$ [Def.15], so $FK = A$. [C.N.1]
$GK = GH$ [Def.15], so $GK = C$. [C.N.1]
And we have already that $FG = B$.

Q.E.F.



Comments:
1. Euclid's proof simply states that the three lines that we are using to construct the triangle comply with the restriction from Proposition I.20, that the sum of the lengths of any two are greater than the length of the third. He does not explain why this is important. We need this result, however, to show that the two circles DKL and HKL intersect at two points. If they did not, we would have no triangle to construct.

2. To prove that the two circles do in fact intersect at two points, we should identify the other point (besides H) where circle HKL intersects DE, and label it J. To be sure that the circles intersect at two points we must prove that $FJ < FD$ and that $FD < FH$:

$B < A + C$, so $FG < FD + GH$. Then $FG - GH < FD$.
But $GH = GJ$, so $FG - GJ < FD$, and so $FJ < FD$.
$A < B + C$, so $FD < FG + GH$, and so $FD < FH$.

Our picture will also look different if the lengths of A, B, and C are not in descending length order, in which case we would need the third part of the condition, that $C < A + B$, in order to prove that the two circles intersect in two points.

3. Euclid's proof also does not quite prove the result as he uses it in later propositions. Here we constructed a triangle, creating the base ourselves. More often, however, Euclid creates a triangle on a base that is already given. For homework we modified this proof to construct the triangle on a given segment.