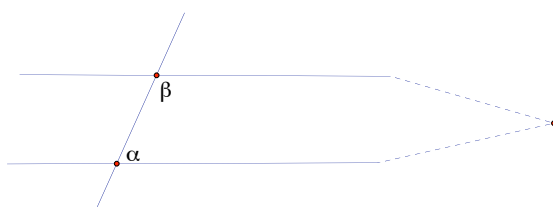


## MT 453 Comments on Postulate 5

Feb. 6 2009

Prop. I.29 is the first use of Post. 5:

*That if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.*



$$\alpha + \beta < \text{┐┐} \Rightarrow \text{lines meet on the side of } \alpha, \beta.$$

The statement of Post. 5 is rather complicated, though it is stated in the form used in *I.29*. At least by the time of Proclus (c. 400 AD) it was known that Post. 5 is equivalent to what is now called “Playfair’s Axiom” after its rediscovery by the Scottish mathematician John Playfair (1748-1819). His version of the *Elements* had three postulates and twelve axioms, of which the eleventh is

### Playfair’s Axiom

*Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.*

In other words:

*Through a point there is at most one line parallel to a given line.*

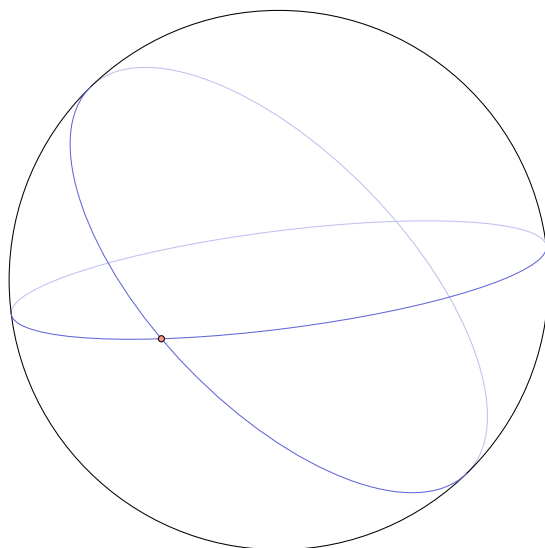
Now Prop. I.31 constructs a line parallel to a given line through a given point. This relies on I.16, which does not use Post. 5. So it is the *uniqueness* of the parallel, not its *existence* that requires Post. 5, hence Playfair’s formulation. Nevertheless, it is interesting to rephrase Playfair’s axiom to include both existence and uniqueness:

### **Revised Playfair's Axiom**

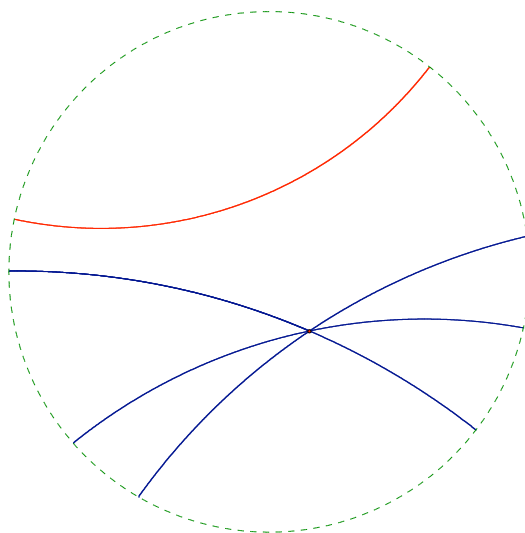
*Through a point there is one and only one line parallel to a given line.*

The two parts of the revised Playfair axiom characterize Euclidean Geometry among the other two-dimensional Geometries.

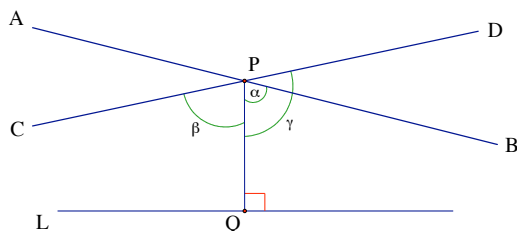
Spherical Geometry. Props. I.1 to I.15 hold. Parallels do not exist:



Hyperbolic Geometry. Props. I.1 to I.28 hold. Parallels exist but are not unique:



Here is a proof that Post. 5 implies Playfair's axiom. (The converse is in your homework.)



Let  $L$  be a line and  $P$  a point.

Suppose  $AB$  and  $CD$  are two distinct lines through  $P$  parallel to  $L$ .

Draw the perpendicular from  $PQ$  from  $P$  to  $L$  [I.12].

Let  $\alpha = \angle BPQ$ ,  $\beta = \angle CPQ$ ,  $\gamma = \angle DPQ$ .

Since  $AB$  is parallel to  $L$ , we have  $\alpha + \perp \geq \perp$ . [Post.5]

Therefore  $\alpha \geq \perp$ . Likewise,  $\beta \geq \perp$ , so  $\alpha + \beta \geq \perp$ .

But  $\beta + \gamma = \perp$  [I.13].

And  $\alpha$  is part of  $\gamma$ , so  $\alpha < \gamma$  [c.n.5].

Hence,  $\beta + \alpha < \perp$ , a contradiction.