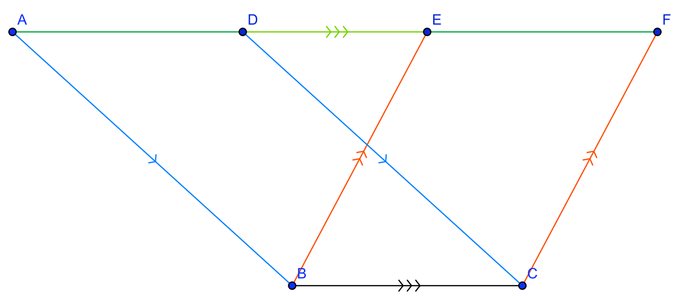
MT 453 Elements Day 11

Speakers: Daniel Moresco
Scribes: Thomas Quan, Rebecca Wentzel

February 11, 2008

Proposition I.35

Parallelograms which are on the same base and in the same parallels are equal to one another.



Let $ABCD$ and $BEFC$ be parallelograms sharing the base BC and in the same parallels AF and BC.

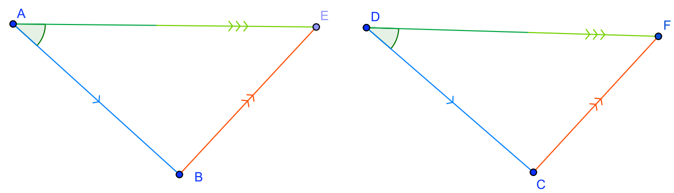
Since opposite sides in a parallelogram are equal,

$AD = BC$ [I.34]

and $BC = EF$ [I.34]

So $AD = EF$ [C.N. 1]

Now focus on $\triangle ABE$ and $\triangle DCF$.



Since $AD = EF$,

$AD + DE = EF + DE$, [C.N. 2]

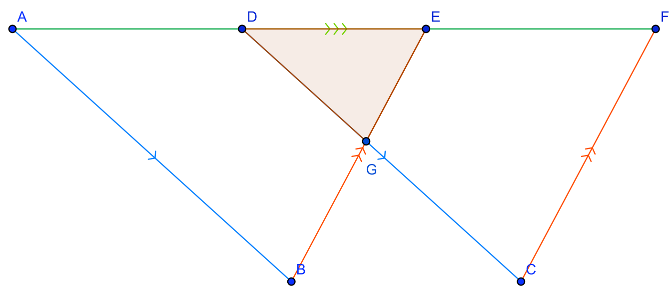
which means $AE = DF$.

$AB = DC$, since AB and DC are opposite sides of a parallelogram. [I.34]

$\angle BAE = \angle CDF$, since $\angle BAE$ and $\angle CDF$ are the interior and exterior angles of a straight line falling on two parallel lines. [I.29]

$\triangle ABE \simeq \triangle DCF$, since $AE = DF$, $\angle BAE = \angle CDF$, and $AB = DC$. [I.4]

This implies $BE = CF$ and the areas of $\triangle ABE$ and $\triangle DCF$ are equal.



Subtract the area DGE from $\triangle ABE$ and $\triangle DCF$. [C.N. 3]

$\triangle ABE = \triangle DCF$

$\triangle ABE - DGE = \triangle DCF - DGE$

$ABGD = GEFC$

We now have two trapezoids $ABGD$ and $GEFC$.

Add the area of BGC to both trapezoids. [C.N. 2]

$ABGD + BGC = GEFC + BGC$

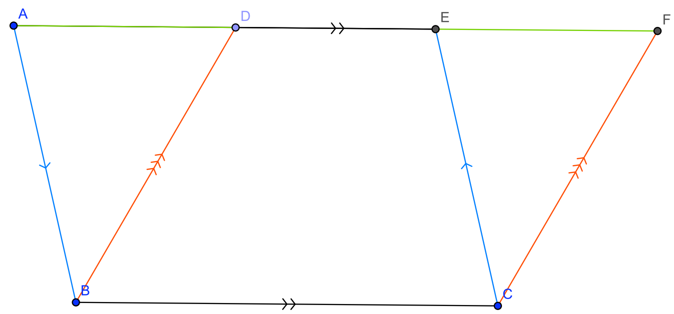
$ABCD = BEFC$.

So the two parallelograms $ABCD$ and $BEFC$ are equal.

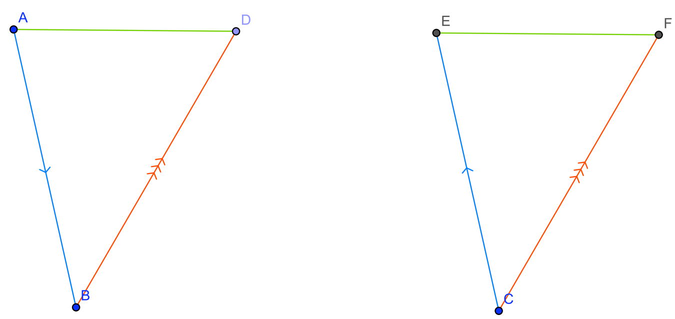
Q.E.D.

Comments:
1. Proposition I.34 could have been used to show that $BE = CF$, since they are opposite sides of the parallelogram $BEFC$. Then Proposition I.7 (Side-Side-Side) could have been used to show that $\triangle ABE \simeq \triangle DCF$.

2. What if the parallelograms were aligned in such away that there was no $\triangle DGE$ (i.e. the tops of the parallelogram touched or overlap)?



Suppose the two parallelograms, $ABCE$ and $BDFC$, overlap at the top of the figure.
Focusing on $\triangle ABD$ and $\triangle ECF$,



we have $AB = EC$ and $DB = FC$ [I.34]
Also $AE = DF$, since $AE = BC$ and $DF = BC$, [I.34] [C.N. 1]
which means $AD = EF$, since $AE = DF \Rightarrow AE - DE = DF - DE$.
This gives us $\triangle ABD \simeq \triangle ECF$ [I.7]
So the areas of the two triangles are equal and adding the area $BDEC$ shows $ABCE = BDFC$.
The case where the tops of the parallelograms just touch is similar.

3. A third case, in which both parallelograms lean in one direction can also be shown to be true. In fact, this is an example of what is called a **dynamic config-**

uration. A dynamic configuration is a configuration, in which something never changes. In our case the area of the parallelogram never changes. Consider a parallelogram, whose base is fixed, but whose opposing side is connected to an indefinite straight line parallel to the base. This opposing side can be moved in either direction and for any distance along the indefinite straight line, but the area of this parallelogram will remain unchanged.