Proposition I.7

Speaker: Sean Johnson
Scribe: Matt MacDonald
Artist: Daniel Moresco

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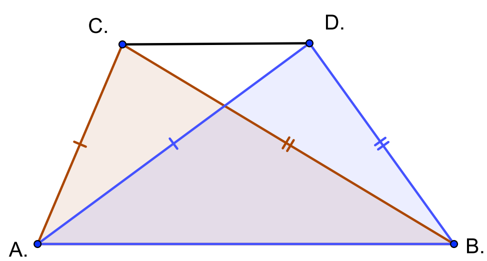
From the extremities of a straight line, it is not possible to construct two sets of straight lines that are equal to each other, respectively, that meet in different points.

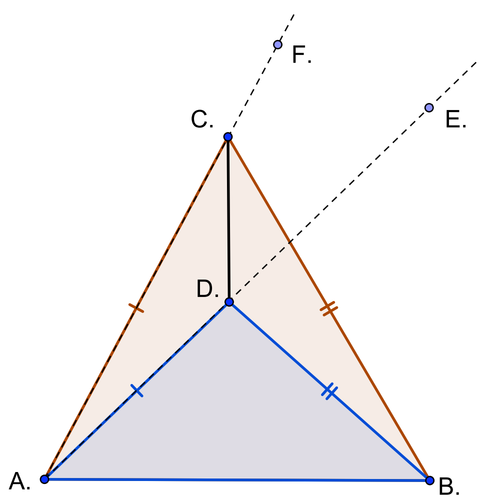


Let AB be the starting straight line, with lines AC and CB constructed on it, meeting at the point C.
Construct AD and DB on the same side of AB such that AD = AC, BC = BD. (Post. 1 and Prop. I.2)
Connect C and D. (c.n. 2)
Since AC = AD, $\angle ACD$ = $\angle ADC$. (Prop I.5)
$\angle ACD$ is greater than angle BCD, as the whole is greater than the part. (c.n. 5)
This means that $\angle ADC$ is greater than $\angle BCD$.
But, since BC = BD, $\angle ACD$ = $\angle BDC$. (Prop I.5)

This is a contradiction, so the original assumption that the lines met in different points is false.
Q.E.F.

Comments: 1. Euclid mentions that the second point must be constructed on the same side as the first point, yet he does not define what he means by side. This is intuitive to us as modern mathematicians, but only if we are dealing with situations in a plane. Although never explicitly stated, it is assumed that these proofs take place in a plane. If we were in a three dimensional space, it would be possible to construct two sets of equal lines meeting in different points using the freedom we have with the lack of a definition of same side.

2. A separate case that Euclid does not discuss is if the triangle formed by the second set of lines is inside the first triangle, resulting in a picture such as this:



This proof would be slightly different, as it would require extending the sides of the isosceles triangles and using the fact that the angles below the bases are equal (Prop I.5) to prove that the two points must coincide.

3. An alternate proof for this proposition involves drawing a circle E with radius AC and a circle F with radius BC (Post. 3). We then suppose we have a point D such that $AD = AC$ and $DB = DC$. This means that D lies on circle E and circle F. Only one point lies on the same side of AB and both circles, and that is C.