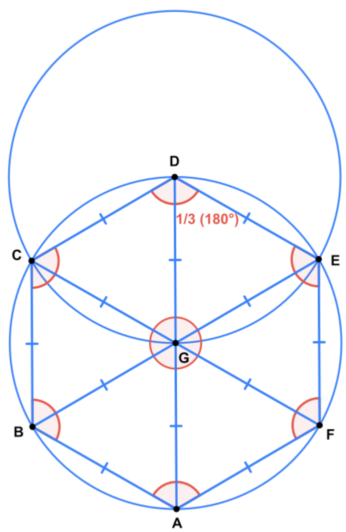
MT 453 Elements Day 24

Speaker: Kerry Fitzmaurice
Scribes: Thomas Quan, Kaitlyn Valente

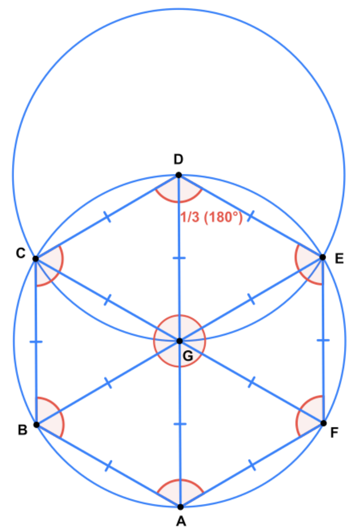
March 18/20, 2009

Proposition IV.15

To inscribe a regular hexagon in a given circle



Let $ABCDEF$ be a circle.
Find the center, G, of $ABCDEF$. [III.1]
Draw the diameter AD.
Draw a circle with center D and radius DG,
where the two circles intersect at C and E.
Draw the diameters CF and BE. [Post. 2]
Draw AB, BC, CD, DE, EF, FA. [Post. 1]



Focus on $\triangle GDE$.

$DG = EG$, since they are radii of the circle centered at G.

$DG = DE$, since they are radii of the circle centered at D.

So $DG = EG = DE$ [C.N. 1]

Therefore $\triangle GDE$ is an equilateral triangle and

$\angle GDE = \angle DEG = \angle EGD$ [I.5]

$\angle GDE + \angle DEG + \angle EGD = \lrcorner\lrcorner$ [I.32]

Therefore, $\angle GDE = \angle DEG = \angle EGD = \frac{1}{3}\lrcorner\lrcorner$

Similarly, $\triangle GCD$ can be shown to be equilateral with angles equal to $\frac{1}{3}\lrcorner\lrcorner$

So, $\angle CGD = \angle DGE = \frac{1}{3}\lrcorner\lrcorner$.

CG is set up on BE so $\angle BGC + \angle CGD + \angle DGE = \lrcorner\lrcorner$. [I.13]

Therefore, $\angle BGC = \frac{1}{3}\lrcorner\lrcorner$.

Because of vertical angles, $\angle DGE = \angle CGD = \angle BGC = \angle AGB = \angle AGF = \angle FGE = \frac{1}{3}\lrcorner\lrcorner$. [I.15]

So $\triangle DGE \simeq \triangle CGD \simeq \triangle BGC \simeq \triangle AGB \simeq \triangle AGF \simeq \triangle FGE$. [I.4]

Therefore the pentagon is equilateral, $AB = BC = CD = DE = EF = FA$.

And since the angles of all six triangles are equal the sum of two of these angles are equal.

Therefore the pentagon is equiangular, $\angle ABC = \angle BCD = \angle CDE = \angle DEF = \angle EFA = \angle FAB$.

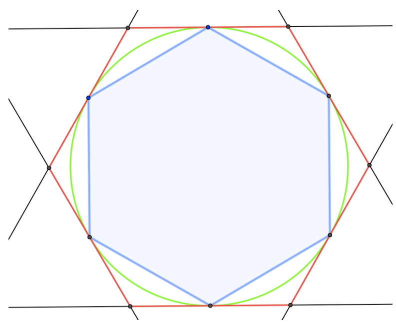
Thus the hexagon is a regular hexagon.

Q.E.D.

Porism:

1. The radius of the exscribed circle is the length of the side of the hexagon.

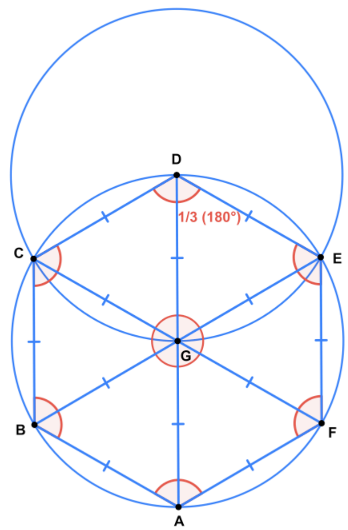
2. A regular hexagon can be exscribed by drawing tangents to the circle at the endpoints of the inscribed hexagon.



3. Given a regular hexagon, a circle can be inscribed and a circle can be exscribed.

Comments:

We could have used arcs (as Euclid does) to prove that the hexagon is equilateral and equiangular rather than congruent triangles .



$\angle DGE = \angle CGD = \angle BGC = \angle AGB = \angle AGF = \angle FGE = \frac{1}{3}\lrcorner\lrcorner$,

But equal angles stand on equal circumferences. [III.26]

Equal circumferences are subtended by equal straight lines. [III.29]

Therefore $AB = BC = CD = DE = EF = FA$, so the hexagon is equilateral.

Since the angles of the hexagon are at the circumference of the circle and stand on equal circumferences they are equal. [III.27]

Therefore $\angle ABC = \angle BCD = \angle CDE = \angle DEF = \angle EFA = \angle FAB$, so the hexagon is equiangular.

Thus the hexagon is regular.

Q.E.D.