MT 453 Elements Day 22

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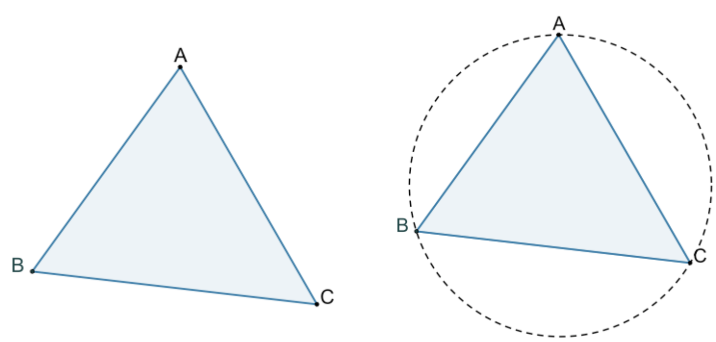
Proposition IV.5

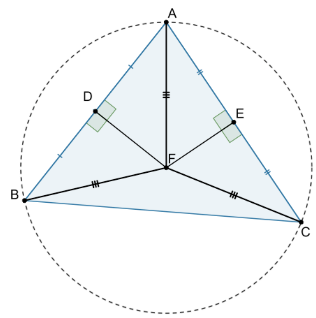
How to circumscribe a circle about a given triangle.

According to Definition IV.6, a circle is circumscribed about a figure if the circumference of the circle passes through each angle of the figure.



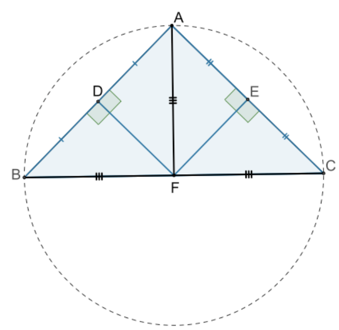
Let $\triangle ABC$ be the given triangle. Bisect AB at point D and AC at E (Prop. I.10). Draw the perpendiculars to AB at D and to AC at E (Prop. I.11), meeting at point F. We distinguish three cases, based on the position of F.

Case 1: F is inside $\triangle ABC$.



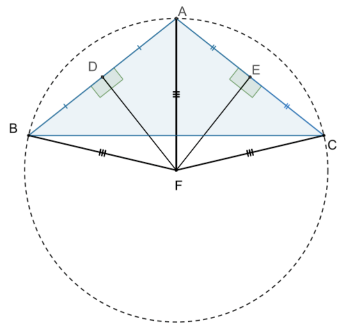
Draw AF, BF, and CF. Then $\triangle ADF \cong \triangle BDF$ (Prop. I.4, SAS), so $AF = BF$. Similarly, $\triangle AEF \cong \triangle CEF$, so $AF = CF$. We conclude that $AF = BF = CF$, and then draw the circle of that radius centered at F.

Case 2: F is on BC.



Draw AF. A similar argument shows that $AF = BF = CF$, so we again draw the circle of that radius centered at F.

Case 3: F is outside$\triangle ABC$.



Draw AF, BF, and CF. Again, $\triangle ADF \cong \triangle BDF$, so $BF = AF$, and $\triangle AEF \cong \triangle CEF$, so $AF = CF$. Draw the circle of radius $AF = BF = CF$ centered at F.

Q.E.F.

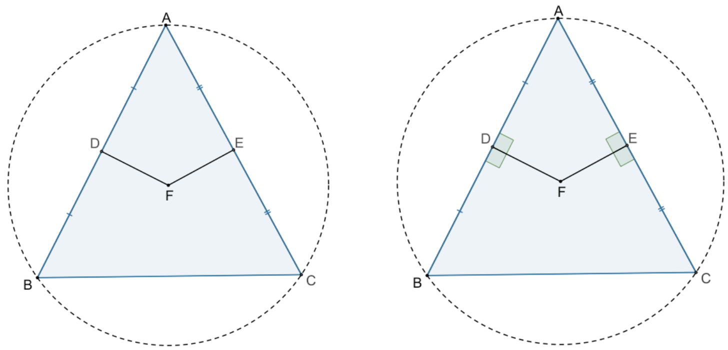
Comments:

1. Only Propositions I.4, I.10, and I.11 are used, so this could have been presented much earlier.

2. Euclid comments that in Case 1, $\angle BAC$ is acute, in Case 2 right, and in Case 3 obtuse.

3. Is the circle unique? Yes. We could appeal to Proposition III.10: *A circle does not cut a circle at more points than two.* Or we could argue as follows:

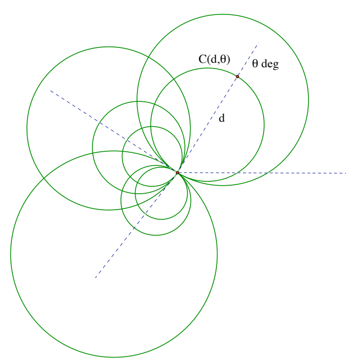
Draw a circumscribing circle with center at F.



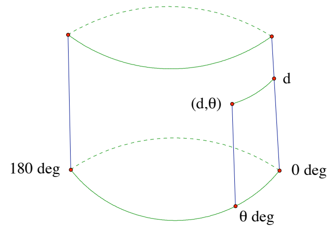
Since E bisects AC, we have that $FE \perp AC$ by Prop. III.3. But similarly, $FD \perp AB$. Thus F, as the intersection of the perpendicular bisectors of two sides of the triangle, is unique.

4. How many circles pass through a given number of points?

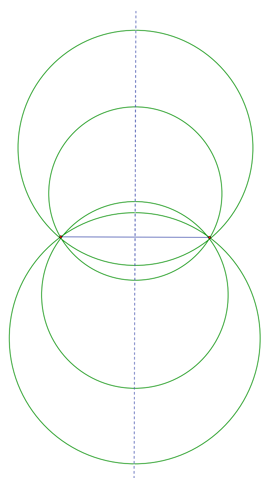
Through one point, we can draw a circle $C(d, \theta)$ of any diameter d at any angle θ: *two dimensions* of choice.



We can visualize these choices as points on a cylinder (a 2-dimensional geometric object):



Through two points, we can draw circles $C(d)$ of any diameter d: *one dimension* of choice.



The choice corresponds to a point on the perpendicular bisector of the segment between the two points (a 1-dimensional geometric object).

By this Proposition, through three noncollinear points, determining a triangle, there is one (circumscribed) circle, corresponding to a point, or *0 dimensions* of choice.

Through four (or more) points, there need not be a circle. To build such a set of points, given three noncollinear points, construct the circle through them, and then the choose a fourth point *not on the circle*.