Chapter 13. VECTORS IN THREE DIMENSIONAL SPACE

Let's begin with some names and notation for things:

$\mathbb{R}$ is the set (collection) of real numbers. We write $x \in \mathbb{R}$ to mean that x is a real number. A real number is also called a **scalar** because it can be used to scale vectors.

$\mathbb{R}^2$ is the usual xy-plane. More precisely, $\mathbb{R}^2$ is the set of vectors $\mathbf{u} = (x, y)$, with $x, y \in \mathbb{R}$. We have studied vectors in $\mathbb{R}^2$ in the previous chapters.

$\mathbb{R}^3$ is three dimensional xyz space. More precisely, $\mathbb{R}^3$ is the set of vectors $\mathbf{u} = (x, y, z)$ with $x, y, z \in \mathbb{R}$. In this chapter we concentrate on vectors in $\mathbb{R}^3$.

There are four vectors in $\mathbb{R}^3$ having special notation:

$$\mathbf{e}_1 = (1, 0, 0), \qquad \mathbf{e}_2 = (0, 1, 0), \qquad \mathbf{e}_3 = (0, 0, 1), \qquad \mathbf{0} = (0, 0, 0).$$

The vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are called the **standard basis vectors** and the vector $\mathbf{0}$ is called the **zero vector**.

Vectors as arrows

Any two points in space determine a vector that measures the difference between the points. If $P = (a, b, c)$ and $P' = (a', b', c')$ are two points in $\mathbb{R}^3$ then the vector $\mathbf{u}$ from P to P' is

$$\mathbf{u} = (a' - a, b' - b, c' - c).$$

You can visualize $\mathbf{u}$ as an arrow drawn from P to P'. If you move this arrow, without changing its length or direction, so that the base of the arrow is at $(0, 0, 0)$, then the tip of the arrow will be at $(a - a', b - b', c - c')$.

Adding vectors

We can do arithmetic with vectors. We'll start with vector addition and related operations. Suppose you have two vectors

$$\mathbf{u} = (x, y, z), \qquad \mathbf{v} = (x', y', z').$$

Then you can add these two vectors:

$$\mathbf{u} + \mathbf{v} = (x + x', y + y', z + z'),$$

subtract them:

$$\mathbf{u} - \mathbf{v} = (x - x', y - y', z - z'),$$

and multiply one of them by a scalar $c \in \mathbb{R}$:

$$c\mathbf{u} = (cx, cy, cz).$$

If you combine these operations with several vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ and scalars $c_1, c_2, \ldots, c_n$, then you get what is called a **linear combination** of the vectors $\mathbf{u}_i$:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n.$$

Every vector in $\mathbb{R}^3$ is a linear combination of the standard basis vectors

$$\mathbf{e}_1 = (1, 0, 0), \qquad \mathbf{e}_2 = (0, 1, 0), \qquad \mathbf{e}_3 = (0, 0, 1).$$

Namely, if $\mathbf{u} = (x, y, z)$ then

$$\mathbf{u} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

If we have two non-proportional vectors $\mathbf{u}$ and $\mathbf{v}$, the collection of all their linear combinations $c_1\mathbf{u} + c_2\mathbf{v}$ forms a plane, called the **span** of $\mathbf{u}$ and $\mathbf{v}$. If we take only those linear combinations where $0 \le c_i \le 1$ for for both $i = 1$ and $i = 2$, we get the **parallelogram** spanned by $\mathbf{u}$ and $\mathbf{v}$

THE DOT PRODUCT

There are two different ways to multiply vectors in $\mathbb{R}^3$. For the first way, take two vectors

$$\mathbf{u} = (x, y, z), \quad \mathbf{v} = (x', y', z').$$

Their **dot product** is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = xx' + yy' + zz'. \tag{1}$$

(In other books, this is sometimes written $\mathbf{u} \cdot \mathbf{v}$, hence the name “dot”. The notation $\langle \mathbf{u}, \mathbf{v} \rangle$ will be more convenient for later calculations.)

The dot product is **symmetric**, meaning that

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle,$$

and **bilinear**, meaning that if you have three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars a, b, then

$$\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle = a\langle \mathbf{u}, \mathbf{w} \rangle + b\langle \mathbf{v}, \mathbf{w} \rangle.$$

If you think of the vector $\mathbf{u} = (x, y, z)$ as an arrow, then the **length** $|\mathbf{u}|$ of $\mathbf{u}$ is given by

$$|\mathbf{u}| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{x^2 + y^2 + z^2}.$$

The vector $\mathbf{u}$ is called a **unit vector** if $|\mathbf{u}| = 1$. This means that $\mathbf{u}$ lies on the unit sphere centered at $\mathbf{0}$ in $\mathbb{R}^3$.

If $\mathbf{u}$ is any nonzero vector, then we can scale it

$$\frac{1}{|\mathbf{u}|}\mathbf{u}$$

to get a unit vector in the same direction.

Geometrically, the dot product of $\mathbf{u}$ and $\mathbf{v}$ is the product of their lengths times the cosine of the angle θ between them. That is,

$$\langle \mathbf{u}, \mathbf{v} \rangle = |\mathbf{u}||\mathbf{v}| \cos \theta. \tag{2}$$

To see this, apply the Law of Cosines to the triangle with vertices $\mathbf{0}, \mathbf{u}, \mathbf{v}$. We find that

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos(\theta).$$

Writing the lengths in terms of dot products via equation (1), this says

$$\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle - 2|\mathbf{u}||\mathbf{v}| \cos \theta.$$

Using symmetry and bilinearity, this equation becomes

$$\langle \mathbf{u}, \mathbf{u} \rangle - 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle - 2|\mathbf{u}||\mathbf{v}| \cos \theta,$$

and if we simplify this, we get equation (2).

Computations with the dot product:

(i) *The angle between two vectors.*

For example, the vectors $\mathbf{u}$ and $\mathbf{v}$ are **perpendicular** (also called **orthogonal**) exactly when

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

This is because $\cos \theta = 0$ exactly when θ is an odd multiple of $\pi/2$.

More generally, you can use equations (1) and (2) to find the angle between any two vectors. For example, suppose we have the vectors

$$\mathbf{u} = (1, 2, 3), \quad \mathbf{v} = (3, 2, 1),$$

then using (1) we get

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3 + 4 + 3 = 10.$$

Then using (2) we get

$$10 = \langle \mathbf{u}, \mathbf{v} \rangle = \sqrt{14}\sqrt{14}\cos\theta,$$

so

$$\cos\theta = \frac{10}{14} = \frac{5}{7}.$$

(ii) *The projection of a vector onto a line*.

Note that if $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, then (2) simplifies:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \cos\theta, \quad \text{if} \quad |\mathbf{u}| = |\mathbf{v}| = 1.$$

If just one of them, say $\mathbf{u}$, is a unit vector, then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \text{length of projection of } \mathbf{v} \text{ onto the line through } \mathbf{u}.$$

(iii) *The equation of a plane*.

Given a plane through a point $P_0 = (x_0, y_0, z_0)$, and a vector $\mathbf{n} = (a, b, c)$ perpendicular to the plane, the plane has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{3}$$

This is because a general point $P = (x, y, z)$ lies on the plane exactly when the vector $\mathbf{u}$ from P_0 to P is perpendicular to $\mathbf{n}$. This happens exactly when $\langle \mathbf{n}, \mathbf{u} \rangle = 0$, which is equation (3). The vector $\mathbf{n}$ is called the **normal vector** of the plane.

(iv) *Orthonormal Bases*.

Suppose we have three vectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3$. We say these vectors form an **orthonormal basis** of $\mathbb{R}^3$ if

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

To see lots of orthonormal bases, take a cube with side length one, and put one corner at $\mathbf{0}$. The three edges coming out of this corner are an orthonormal basis.

For example, the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is orthonormal. Taken in this order, the standard basis is right handed, in the sense that if you bend your right hand to a right angle, such that the first vector (in this case $\mathbf{e}_1$) points along your fingers toward the tips, and the second vector (in this case $\mathbf{e}_2$) points along your palm toward the wrist, then your thumb will point in the direction of the third vector (in this case $\mathbf{e}_3$). On the other hand (!) the orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3$ is left handed. Every orthonormal basis is either left-handed or right-handed. The right-handed ones are obtained by simultaneously rotating $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and the left handed ones by simultaneously rotating $\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3$.

THE CROSS PRODUCT IN $\mathbb{R}^3$

The second way of multiplying vectors in $\mathbb{R}^3$ gives you another vector. Take two vectors

$$\mathbf{u} = (x, y, z), \quad \mathbf{v} = (x', y', z').$$

Their cross product is the vector

$$\mathbf{u} \times \mathbf{v} = (yz' - zy', zx' - xz', xy' - yx'). \tag{4}$$

One way to remember this is via the 2×3 matrix

$$\begin{bmatrix} x & y & z \\ x' & y' & z' \end{bmatrix}.$$

Let A_i be the 2×2 submatrix obtained by removing the i^{th} column:

$$A_1 = \begin{bmatrix} y & z \\ y' & z' \end{bmatrix}, \qquad A_2 = \begin{bmatrix} x & z \\ x' & z' \end{bmatrix}, \qquad A_3 = \begin{bmatrix} x & y \\ x' & y' \end{bmatrix}.$$

Then

$$\mathbf{u} \times \mathbf{v} = (\det A_1, -\det A_2, \det A_3).$$

Basic properties of cross-product:

1. (Bilinearity) If you have three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars a, b, then

$$(a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = a(\mathbf{u} \times \mathbf{w}) + b(\mathbf{v} \times \mathbf{w}).$$

2. (Antisymmetry)
$$\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}.$$
In particular, we have
$$\mathbf{u}\times\mathbf{u} = \mathbf{0}.$$
In fact, we have $\mathbf{u}\times\mathbf{v} = 0$ exactly when $\mathbf{u}$ and $\mathbf{v}$ are proportional.

3. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

4. $\mathbf{u}\times\mathbf{v}$ points in the direction of your right thumb when $\mathbf{u}$ points along your fingers and $\mathbf{v}$ points along your palm toward the wrist.

5. The length of $|\mathbf{u}\times\mathbf{v}|$ is given by
$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta, \tag{5}$$
where θ is the acute angle (between 0 and π) between $\mathbf{u}$ and $\mathbf{v}$.

You will verify some of these properties in the exercises below.

Computations with the cross-product:

(i) *Area of a triangle in space*.

Equation means that $|\mathbf{u}\times\mathbf{v}|$ is the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. Hence the triangle with edges $\mathbf{u}, \mathbf{v}, \mathbf{v}-\mathbf{u}$ has area $\frac{1}{2}|\mathbf{u}\times\mathbf{v}|$.

Here's another way to think of this area: Take a triangle Δ in space and let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be the vertices of Δ. If you think of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as arrows based at $\mathbf{0}$, then Δ is formed by the tips of these arrows.

Two of the sides of Δ are the vectors $\mathbf{v}-\mathbf{u}$ and $\mathbf{w}-\mathbf{u}$. So the area of Δ is
$$\mathrm{Area}(\Delta) = \tfrac{1}{2}|(\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u})|.$$
Using bilinearity of the cross product, we get
$$(\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u}) = \mathbf{v}\times\mathbf{w} - \mathbf{u}\times\mathbf{w} - \mathbf{v}\times\mathbf{u} + \mathbf{u}\times\mathbf{u}.$$
By antisymmetry, we have $\mathbf{u}\times\mathbf{u} = 0$, and
$$(\mathbf{v}-\mathbf{u})\times(\mathbf{w}-\mathbf{u}) = \mathbf{v}\times\mathbf{w} + \mathbf{w}\times\mathbf{u} + \mathbf{u}\times\mathbf{v}.$$

So we get the formula for the area of a triangle Δ with vertices $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$\text{Area}(\Delta) = \frac{1}{2}|\mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} + \mathbf{w} \times \mathbf{u}|. \tag{6}$$

Note the cyclic pattern. You will use equation (6) to prove the three dimensional Pythagorean theorem in Exercise 13.7 below.

(ii) *Equation of the plane spanned by two vectors.*

Recall that two nonproportional vectors $\mathbf{u}$ and $\mathbf{v}$ span a plane through the origin. The vectors on this plane are all the linear combinations $c_1\mathbf{u} + c_2\mathbf{v}$. Since $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$, it is it is also perpendicular to each linear combination $c_1\mathbf{u} + c_2\mathbf{v}$. Hence $\mathbf{u} \times \mathbf{v}$ is normal to the plane. Let's say that $\mathbf{u} \times \mathbf{v} = (a, b, c)$. Then the equation of the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ is

$$ax + by + bz = 0.$$

(iii) *Intersection of two planes.*

Take two distinct planes through the origin. Say they have equations

$$ax + by + cz = 0, \qquad a'x + b'x + c'x = 0.$$

Their normal vectors are

$$\mathbf{n} = (a, b, c) \qquad \text{and} \quad \mathbf{n}' = (a', b', c'),$$

respectively. Since the planes are distinct, these vectors are nonproportional. The vector $\mathbf{n} \times \mathbf{n}'$ is nonzero and perpendicular to both $\mathbf{n}$ and $\mathbf{n}'$, so it lies on both planes. The line of intersection of the planes is therefore the line through $\mathbf{n} \times \mathbf{n}'$.

Exercise 13.1 Let $\mathbf{u} = (x, y, z)$, $\mathbf{v} = (x', y', z')$. Use the dot product to verify that $\langle \mathbf{u}, \mathbf{u} \times \mathbf{v} \rangle = 0$.

Exercise 13.2 Let $\mathbf{u} = (1, -2, 1)$, $\mathbf{v} = (2, 1, 1)$. Find $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\mathbf{u} \times \mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$.

Exercise 13.3 Find a nonzero vector on the line of intersection of the two planes

$$x - 2y + z = 0, \quad 2x + y + z = 0.$$

Exercise 13.4 Find the angle between the diagonal of a cube and one of the edges of the cube. It doesn't matter which cube you use, but you may wish to use the cube whose vertices are the eight points (x, y, z) with x, y, z either 0 or 1.

Exercise 13.5 Take two vectors

$$\mathbf{u} = (x, y, z) \quad \text{and} \quad \mathbf{v} = (x', y', z').$$

Since $\mathbf{u} \times \mathbf{v}$ is a vector, its length $|\mathbf{u} \times \mathbf{v}|$ is a number. The dot product $\langle \mathbf{u}, \mathbf{v} \rangle$ is also a number. The relation between these numbers is:

$$\langle \mathbf{u}, \mathbf{v} \rangle^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2. \tag{7}$$

Prove equation (7) by expanding both sides in terms of x, y, z, x', y', z'.

Exercise 13.6 Use Equations (2) and (7) to prove the length formula

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$

Exercise 13.7 Consider the tetrahedron with vertices

$$A = (a, 0, 0), \quad B = (0, b, 0), \quad C = (0, 0, c), \quad \text{and} \quad O = (0, 0, 0).$$

For any three points X, Y, Z in space, write $[XYZ]$ for the area of the triangle with vertices X, Y, Z. Use Equation (6) to prove that

$$[ABC]^2 = [OAB]^2 + [OBC]^2 + [OCA]^2.$$

This is the **three dimensional Pythagorean theorem**.