Chapter 14. Three-by-Three Matrices and Determinants

A 3×3 matrix looks like

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [a_{ij}]$$

The number a_{ij} is the entry in row i and column j of A. Note that A has three **row vectors**:

$$\text{row}_i(A) = (a_{i1}, a_{i2}, a_{i3}), \qquad i = 1, 2, 3$$

as well as three **column vectors**:

$$\text{col}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix}, \qquad j = 1, 2, 3.$$

The product of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is the matrix $AB = [c_{ij}]$ whose entry in row i and column j is the dot product

$$c_{ij} = \langle \text{row}_i(A), \text{col}_j(B) \rangle = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}.$$

The identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It has the property that $AI = IA = A$ for every 3×3 matrix A. The **inverse** of a matrix A is the matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

As with the 2×2 case, not all matrices are invertible. Later, we will see exactly when A^{-1} exists, and we will have a formula for A^{-1} when it does exist. Finally, the **transpose** of A is the matrix A^T such that $\text{row}_i(A^T) = \text{col}_i(A)$. That is, to get A^T you flip A about the diagonal.

A 3×3 matrix A moves vectors around in space. More precisely, A can be viewed as the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which sends the vector $\mathbf{u} = (x, y, z)$ to the vector

$$A\mathbf{u} = \begin{bmatrix} \langle \text{row}_1(A), \mathbf{u} \rangle \\ \langle \text{row}_2(A), \mathbf{u} \rangle \\ \langle \text{row}_3(A), \mathbf{u} \rangle \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix}.$$

Recall that the standard basis vectors in $\mathbb{R}^3$ are

$$\mathbf{e}_1 = (1,0,0), \qquad \mathbf{e}_2 = (0,1,0), \qquad \mathbf{e}_3 = (0,0,1).$$

The Most Important Thing, again, is that the j^{th} column of A is what A does to $\mathbf{e}_j$. In other words,

$$\boxed{\mathrm{col}_j(A) = A\mathbf{e}_j} \qquad \text{(M.I.T.)}$$

Example: The matrix

$$A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

is a rotation by θ around the y axis. You can see right away that the y axis is fixed, since

$$A\mathbf{e}_2 = \mathbf{e}_2. \tag{1}$$

The corner entries of A give a rotation in the xz plane. Since for small θ the z-coordinate of $A\mathbf{e}_1$ is positive, the rotation is clockwise by θ, as the positive y-axis points towards the viewer.

Example: The matrix

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is again a rotation matrix, but now the axis is not so obvious. First note that

$$B\mathbf{e}_1 = \mathbf{e}_2, \qquad B\mathbf{e}_2 = \mathbf{e}_3, \qquad B\mathbf{e}_3 = \mathbf{e}_1$$

so that B permutes the vertices of the frst quadrant unit cube (whose vertices have components either 0 or 1). Note also that that $B^3 = I$ and that B fixes the vector $\mathbf{u} = (1,1,1)$:

$$B\mathbf{u} = \mathbf{u}, \qquad \text{where} \quad \mathbf{u} = (1,1,1). \tag{2}$$

Now you can see that B is rotation by $2\pi/3$ about the diagonal of the cube through the opposite vertices **0** and $\mathbf{u}$.

In both of these examples, the axis of rotation is through an eigenvector with eigenvalue $\lambda = 1$, as expressed in equations (1) and (2). To find eigenvalues and eigenvectors, we need some computational tools. The main tool is the **determinant**, which you can compute as follows.

If A is a 3×3 matrix, we define A_{ij} to be the 2×2 submatrix of A obtained by deleting row i and column j. For example, if $i = 1$ and $j = 3$, then

$$A_{13} = \begin{bmatrix} \cdot & \cdot & \cdot \\ a_{21} & a_{22} & \cdot \\ a_{31} & a_{32} & \cdot \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

The determinant of A is defined to be

$$\boxed{\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}} \qquad (3)$$

Note the connnection with the cross-product: If we write

$$A = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix},$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are the row vectors of A, then

$$\det(A) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}). \qquad (4)$$

Formula (4) also holds when $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$, that is, when $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are the column vectors of A.

Formula (3) is an expansion along the top row. In fact, you can expand along any row or column. For example, you can also compute $\det A$ as

$$\det A = -a_{21} \det A_{21} + a_{22} \det A_{22} - a_{23} \det A_{23}, \quad \text{(second row)}$$

$$\det A = a_{13} \det A_{13} - a_{23} \det A_{23} - a_{33} \det A_{33}, \quad \text{(third column)}.$$

The rule for the signs is given in the following picture:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

That is, the term $a_{ij} \det A_{ij}$ gets sign $(-1)^{i+j}$.

No matter which row or column you choose to expand, the result will be

$$\begin{aligned} &\det A = \\ &a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}. \end{aligned} \qquad (5)$$

Basic Properties of the Determinant:

1. $\det(AB) = (\det A)(\det B)$
2. $\det(A) = \det(A^T)$.
3. A matrix A has an inverse A^{-1} exactly when $\det(A) \neq 0$, in which case
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$
4. If you switch two rows of A, the determinant changes by a sign. The same holds for the columns. For example,
$$\det \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \\ \mathbf{v} \end{bmatrix} = -\det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} \mathbf{u} & \mathbf{w} & \mathbf{v} \end{bmatrix} = -\det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$$
5. If you add a multiple of one row to another row the determinant is unchanged. The same holds for the columns. For example,
$$\det \begin{bmatrix} \mathbf{u} + c\mathbf{w} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \det \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix}.$$
6. $|\det(A)|$ is the volume of the parallelopiped spanned by the row (or column) vectors of A.

A **parallelopiped** is a six-sided box with opposite slides parallel. The parallelogram spanned by three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the collection of points
$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}, \quad \text{where } 0 \leq c_i \leq 1 \quad \text{for } \ i = 1, 2, 3.$$
Property 6 of the determinant comes directly from Equation (4), which tells us that
$$|\det A| = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{v} \times \mathbf{w}||\mathbf{u}||\cos\theta|,$$
where θ is the angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$. Now $|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram P spanned by $\mathbf{v}$ and $\mathbf{w}$, and P can be taken as the base of the parallelopiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$. The volume of the latter is the base times the height, and the height is $|\mathbf{u}||\cos\theta|$, proving that $|\det A|$ is the asserted volume.

Formula for the inverse of a matrix:

The determinant can be used to find the inverse of a matrix. Assume that $\det(A) \neq 0$. Let $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3$ be the rows of A. Take cross-products as follows:

$$\mathbf{v}_1 = \mathbf{u}_2 \times \mathbf{u}_3, \qquad \mathbf{v}_2 = \mathbf{u}_3 \times \mathbf{u}_1, \qquad \mathbf{v}_3 = \mathbf{u}_1 \times \mathbf{u}_2,$$

and let B be the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. So we have

$$A = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}, \qquad B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}.$$

Since

$$\det(A) = \langle \mathbf{u}_1, \mathbf{u}_2 \times \mathbf{u}_3 \rangle = \langle \mathbf{u}_3, \mathbf{u}_1 \times \mathbf{u}_2 \rangle = \langle \mathbf{u}_2, \mathbf{u}_3 \times \mathbf{u}_1 \rangle$$

and $\langle \mathbf{u}_i, \mathbf{u}_j \times \mathbf{u}_k \rangle = 0$ if $i = j$ or $i = k$, we have $AB = \det(A)I$. Hence we get the formula

$$A^{-1} = \frac{1}{\det A} B. \tag{6}$$

Of course, this only makes sense if $\det(A) \neq 0$. If $\det(A) = 0$ then A^{-1} does not exist. If you write out Formula (6) in terms of the original row vectors $\mathbf{u}_i$, it looks like

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \mathbf{u}_2 \times \mathbf{u}_3 & \mathbf{u}_3 \times \mathbf{u}_1 & \mathbf{u}_1 \times \mathbf{u}_2 \end{bmatrix}.$$

Note the cyclic pattern again.

Example:

$$A = \begin{bmatrix} 1 & 7 & 9 \\ 2 & 6 & 5 \\ 4 & 8 & 3 \end{bmatrix}.$$

We compute:

$$\mathbf{u}_2 \times \mathbf{u}_3 = (-22, 14, -8), \quad \mathbf{u}_3 \times \mathbf{u}_1 = -(-51, 33, -20), \quad \mathbf{u}_1 \times \mathbf{u}_2 = (-19, 13, -8),$$

and $\det(A) = \langle \mathbf{u}_1, \mathbf{u}_2 \times \mathbf{u}_3 \rangle = 4$, so

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -22 & 51 & -19 \\ 14 & -33 & 13 \\ -8 & 20 & -8 \end{bmatrix}.$$

Exercise 14.1 Find the matrices R_x, R_y, R_z that rotate by π about the x, y, z axes respectively.

Exercise 14.2 Find the matrices R_{xy}, R_{yz}, R_{zx} that reflect about the xy, yz, zx planes, respectively.

Exercise 14.3 Find the determinants of the following matrices. Do not use a calculator and show all of your work.

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ **(c)** $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(d) $A = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}$ **(e)** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **(f)** $A = \begin{bmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{bmatrix}$.

Exercise 14.4 Make up two vectors $\mathbf{u},\ \mathbf{v}$ in $\mathbb{R}^3$, choose two scalars a and b, and let $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. Compute the determinant of the matrix A whose columns are $\mathbf{u}, \mathbf{v}, \mathbf{w}$. The answer does not depend on your choices.

Exercise 14.5 Draw the parallelopiped spanned by $\mathbf{e}_1,\ \mathbf{e}_1+\mathbf{e}_2,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3$. Then compute its volume.

Exercise 14.6 Find the inverses of the following matrices. Do not use a calculator and show all of your work.

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ **(c)** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{bmatrix}$.

Exercise 14.7 There are six ways to permute the numbers $1, 2, 3$. For each permutation σ, let A_σ be the matrix which sends $\mathbf{e}_i$ to $\mathbf{e}_{\sigma(i)}$.

(a) Write down the six matrices A_σ and compute their determinants.

(b) Each permutation σ corresponds to the term $a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)}$ in the expanded determinant (5). What is the relation between the sign of this term and $\det(A_\sigma)$?

(c) Each permutation corresponds to a symmetry of an equilateral triangle with vertices labelled $1, 2, 3$. Three of the six symmetries are rotations of the triangle, and the other three are reflections. What is the relation between this dichotomy and $\det(A_\sigma)$?