Chapter 15. The Kernel of a Three-by-Three Matrix

Our next tool, finding the kernel of a matrix, is one of the most basic calculations in Linear Algebra. Finding eigenvectors is just one of its many uses. By definition, the **kernel** of a matrix A, denoted $\ker(A)$, is the set of vectors which are sent to the zero vector by A. That is,

$$\boxed{\ker(A) = \{\mathbf{u} \in \mathbb{R}^3 : \ A\mathbf{u} = \mathbf{0}\}} \tag{1}$$

Note that if c is any nonzero scalar, then

$$\ker(cA) = \ker(A). \tag{2}$$

Often the matrix A will look better (eg, fewer minus signs, or fractions) if we multiply A by a scalar; Equation (2) says that this does not change the kernel.

If a_{ij} are the entries of A, then a vector $\mathbf{u} = (x, y, z)$ belongs to $\ker(A)$ exactly when the numbers x, y, z satisfy the three equations

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 0 \\ a_{21}x + a_{22}y + a_{23}z &= 0 \\ a_{31}x + a_{32}y + a_{33}z &= 0. \end{aligned} \tag{3}$$

So, finding $\ker A$ amounts to finding the solutions of the above system of equations. We say $\ker A$ is **trivial** or write $\ker A = \mathbf{0}$ if the only vector in $\ker A$ is the zero vector $\mathbf{0} = (0, 0, 0)$. In other words, $\ker A$ is trivial exactly when the only solution to the system of equations (3) is the obvious one, namely $(x, y, z) = (0, 0, 0)$.

In general, there are four sizes of kernels, according to their dimension:

$\ker A = \mathbf{0}$	$dimension = 0$
$\ker A =$ a line through $\mathbf{0}$	$dimension = 1$
$\ker A =$ a plane through $\mathbf{0}$	$dimension = 2$
$\ker A = \mathbb{R}^3$	$dimension = 3.$

The first step to finding $\ker A$ is to determine if it is trivial or not. This is done by the determinant:

$$\boxed{\ker A = \mathbf{0} \qquad \text{exactly when} \qquad \det(A) \neq 0.} \tag{4}$$

This says two things: (i) If $\det(A) \neq 0$ then $\ker A = \mathbf{0}$, and (ii) If $\det(A) = 0$ then $\ker A \neq \mathbf{0}$.

To prove (i), assume that $\det(A) \neq 0$. Then we know from the previous section that A^{-1} exists. If $\mathbf{u}$ is any vector in $\ker A$ then by definition we have

$$A\mathbf{u} = \mathbf{0}.$$

We apply A^{-1} to both sides of this and get

$$\mathbf{u} = A^{-1}A\mathbf{u} = A^{-1}\mathbf{0} = \mathbf{0}.$$

This shows that $\mathbf{u} = \mathbf{0}$. Hence, the only vector in $\ker A$ is $\mathbf{0}$. This means that $\ker A$ is trivial, proving (i).

For the second assertion in (4), we assume $\det(A) = 0$ and will show that $\ker A \neq \mathbf{0}$ by giving a recipe for computing it. There are three possibilities:

- A is the zero matrix (that is, all entries of A are zero). Then $\ker A = \mathbb{R}^3$.

- A is not the zero matrix, but all rows are proportional to each other. Then $\ker A$ is the plane with equation $ax + by + cz = 0$, where (a, b, c) is any nonzero vector proportional to all the rows of A. This is because A looks like
$$A = \begin{bmatrix} ra & rb & rc \\ sa & sb & sc \\ ta & tb & tc \end{bmatrix},$$
for some scalars r, s, t, at least one of which is nonzero. It is then easy to see that $u = (x, y, z)$ is sent to $(0, 0, 0)$ by A exactly when $ax+by+cz = 0$.

- A has at least two non-proportional row vectors, call them $\mathbf{u}$ and $\mathbf{v}$. Then $\ker A$ is the line through $\mathbf{u} \times \mathbf{v}$. Take two nonproportional rows $\mathbf{u}_i, \mathbf{u}_j$, and suppose $\mathbf{u}_k$ is the remaining row. Then the cross product $\mathbf{u} = \mathbf{u}_i \times \mathbf{u}_j$ is nonzero and the three entries of $A\mathbf{u}$ are
$$\langle \mathbf{u}_i, \mathbf{u}_i \times \mathbf{u}_j \rangle, \quad \langle \mathbf{u}_j, \mathbf{u}_i \times \mathbf{u}_j \rangle, \quad \langle \mathbf{u}_k, \mathbf{u}_i \times \mathbf{u}_j \rangle.$$
The first two of these are always zero, and the last is $\pm\det(A)$, which is zero as well, since we are assuming $\det(A) = 0$.

Example 1:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

This matrix is nonzero, and all rows are proportional to $(1, -1, 2)$, so $\ker A$ is the plane with equation $x - y + 2z = 0$.

Example 2:

$$A = \begin{bmatrix} 2 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

The first two rows are proportional, so we know right away that $\det A = 0$. The last two rows are not proportional, so we compute

$$\mathbf{u} = (2 - 2, 2 - 1, 1 - 2) = (0, 1, -1),$$

and $\ker A$ is the line through $\mathbf{u}$. We can check our result:

$$\begin{bmatrix} 2 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example 3:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

Here no two rows are proportional, so we have to compute the determinant. We can expand along any row. Using the first row, we get

$$\det A = 1(-4) - 3(-3) + 5(-1) = 0.$$

So $\ker A$ is the line through $(-4, --3, -1) = (-4, 3, -1)$. (Check it.)

Example 4:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 2 \\ 1 & 1 & -1 \end{bmatrix}.$$

This time,

$$\det A = 1(-4) - 3(-3) + 4(-1) = 1 \neq 0,$$

so $\ker A = \mathbf{0}$.

Example 5: It turns out (as we'll see later) that the matrix

$$A = \frac{1}{7}\begin{bmatrix} -2 & 6 & 3 \\ 3 & -2 & 6 \\ 6 & 3 & -2 \end{bmatrix}$$

is a rotation. Let us find its axis of rotation. A vector $\mathbf{u}$ on this axis will satisfy the equation $A\mathbf{u} = \mathbf{u}$, which can be written as $(I - A)\mathbf{u} = \mathbf{0}$. So our problem boils down to finding the kernel of the matrix

$$I - A = \frac{1}{7}\begin{bmatrix} 9 & -6 & -3 \\ -3 & 9 & -6 \\ -6 & -3 & 9 \end{bmatrix}.$$

The cross-product of the first two rows is a scalar times $(1, 1, 1)$. So A is a rotation about the axis through $(1, 1, 1)$.

Exercise 15.1 Find the kernels of the following matrices

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ **(c)** $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **(e)** $A = \begin{bmatrix} 8 & 15 & 7 \\ 2 & 9 & 5 \\ -10 & -3 & 1 \end{bmatrix}$ **(f)** $A = \begin{bmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{bmatrix}$.

Exercise 15.2 Let

$$A_t = \begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{bmatrix}.$$

(a) Compute $\det A_t$, and factor it. (Note that $\det A_t$ is a polynomial in t.)

(b) For each root $t = \lambda$ of $\det A_t$, find $\ker A_\lambda$.

(c) What is the relation between the dimension of $\ker A_\lambda$, and the multiplicity of λ as a root of $\det A_t$?

Exercise 15.3 Let $\mathbf{x} = (a_1, a_2, a_3)$ and $\mathbf{y} = (b_1, b_2, b_3)$ be two nonzero vectors, and let A be the matrix whose ij entry is $a_{ij} = a_i b_j$. Find the kernel of A.

Exercise 15.4 Let A and B be the following rotation matrices

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}.$$

It turns out that AB is again a rotation matrix. Find the axis of rotation of AB. Simplify the answer as much as you can.

Hint: As in Example 5, the axis is the kernel of $AB - I$.