Chapter 16. Three-by-Three Eigenvalues and Eigenvectors

Take a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

As in the 2×2 case, we want to find numbers λ and nonzero vectors $\mathbf{u}$ in $\mathbb{R}^3$ satisfying

$$A\mathbf{u} = \lambda\mathbf{u}.$$

As before, the number λ is called an **eigenvalue** of A, and $\mathbf{u}$ is a λ**-eigenvector**.

In the 2×2 case, we were really interested in the line containing an eigenvector, the λ-eigenline. In the 3×3 case, there are two interesting possibilities: we could have λ-eigenlines or λ-eigenplanes.

For a given λ, the λ-**eigenspace** is the set $E(\lambda)$ of all vectors $\mathbf{u}$ such that $A\mathbf{u} = \lambda\mathbf{u}$. That is,

$$E(\lambda) = \{\mathbf{u} \in \mathbb{R}^3 : \ A\mathbf{u} = \lambda\mathbf{u}\}.$$

The eigenspace $E(\lambda)$ will be nonzero exactly when λ is an eigenvalue of A. So how do we find the eigenvalues? Again the determinant does the job.

The equation $A\mathbf{u} = \lambda\mathbf{u}$ can be written

$$(\lambda I - A)\mathbf{u} = \mathbf{0}.$$

Therefore the λ-eigenspace of A is the kernel of $\lambda I - A$:

$$E(\lambda) = \ker(\lambda I - A).$$

Since eigenvectors must be nonzero, we need to have $\ker(\lambda I - A) \neq \mathbf{0}$. From the previous section, we know that $\ker(\lambda I - A)$ is nonzero exactly when

$$\det(\lambda I - A) = 0.$$

Therefore, λ must be a root of the **characteristic polynomial**

$$P_A(x) = \det(xI - A).$$

Writing out this polynomial explicitly, we get

$$P_A(x) = x^3 - (a_{11} + a_{22} + a_{33})x^2 + (\det A_{11} + \det A_{22} + \det A_{33})x - \det A.$$

Here a_{ii} are the diagonal entries of A, and A_{ii} are the 2×2 submatrices obtained by deleting the diagonal entries. The characteristic polynomial is easy to remember once we observe the following pattern:

The **trace** of A is

$$\mathrm{tr}(A) = a_{11} + a_{22} + a_{33}.$$

You could think of $\mathrm{tr}(A)$ as the sum of the diagonal 1×1 determinants of A. In general, the coefficient of x^{3-k} in $P_A(x)$ is $(-1)^k$ times the sum of the diagonal $k \times k$ determinants.

Since $P_A(t)$ is a polynomial of degree three, it has at most three roots. Hence a 3×3 matrix has at most three eigenvalues. If λ occurs just once as a root of $P_A(x)$, then $E(\lambda)$ is a line. If λ occurs twice, then $E(\lambda)$ is a line or a plane. If λ occurs thrice, then $E(\lambda)$ could be a line, a plane, or all of $\mathbb{R}^3$. To summarize:

- The eigenvalues of A are the roots λ of the characteristic polynomial

$$P_A(x) = x^3 - \mathrm{tr}(A)x^2 + (\det A_{11} + \det A_{22} + \det A_{33})x - \det A.$$

- For each root λ of $P_A(x)$, the λ-eigenspace $E(\lambda)$ is $\ker(\lambda I - A)$. You can compute this kernel using the methods of chapter 15.

- The dimension of $E(\lambda)$ is at most the multiplicity of λ as a root of $P_A(x)$. In particular, if λ occurs just once as a root of $P_A(x)$, then $E(\lambda)$ is a line.

Example 1:

$$A = \begin{bmatrix} 7 & -12 & 4 \\ 4 & -9 & 4 \\ 4 & -12 & 7 \end{bmatrix}.$$

We compute

$$P_A(x) = x^3 - 5x^2 + 3x + 9.$$

We try to find a root of $P_A(x)$. The only integer roots must be divisors of 9, namely $\pm 1, \pm 3, \pm 9$. We try $x = 1$, it fails. We try $x = -1$ and it works. Factoring out $x + 1$, we get

$$P_A(x) = (x+1)(x^2 - 6x + 9) = (x+1)(x-3)^2.$$

so the eigenvalues are $-1, 3, 3$. The -1 -eigenspace is

$$E(-1) = \ker(-I - A) = \ker(I + A) = \ker \begin{bmatrix} 8 & -12 & 4 \\ 4 & -8 & 4 \\ 4 & -12 & 8 \end{bmatrix}.$$

The rows are not proportional to a single vector, so this kernel is a line, as it should be, since -1 occurs just once as a root of $P_A(x)$. Taking the cross-product of the last two rows of $I + A$, we get

$$(-16, -16, -16) = -16(1, 1, 1),$$

so $E(-1)$ is the line through $(1, 1, 1)$. Next we compute

$$E(3) = \ker(A - 3) = \ker \begin{bmatrix} 4 & -12 & 4 \\ 4 & -12 & 4 \\ 4 & -12 & 4 \end{bmatrix}.$$

Now all rows are proportional to the vector $(1, -3, 1)$, so $E(3)$ is the plane with equation

$$x - 3y + z = 0.$$

In this example, the dimension of $E(\lambda)$ equals the multiplicity of λ as a root of $P_A(x)$ in all cases.

Example 2:

$$A = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

We compute

$$P_A(x) = x^3 - x^2 - x + 1 = (x - 1)^2(x + 1),$$

$$E(1) = \ker(I - A) = \ker \frac{1}{3} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

So $E(1)$ is the plane with equation $x + y + z = 0$. Next,

$$E(-1) = \ker(-I - A) = \ker \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Taking the cross-product of two rows, we find that $E(-1)$ is the line through $\mathbf{u} = (1, 1, 1)$. Note that $\mathbf{u}$ is perpendicular to the plane $E(1)$. This means that A is a reflection about the plane $E(1)$. Again, each eigenspace $E(\lambda)$ has dimension equal to the multiplicity of λ as a root of $P_A(x)$.

Example 3:

$$A = \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix},$$

where $ab \neq 0$ and c is arbitrary. We compute

$$P_A(x) = x^3,$$

so the only eigenvalue is $\lambda = 0$, with multiplicity three. Since $ab \neq 0$, the first two rows are not proportional and their cross-product is $(ab, 0, 0)$, so $E(0) = \ker A$ is the line through $\mathbf{e}_1 = (1, 0, 0)$. Note that the dimension of $E(0)$ is smaller than the multiplicity of 0 as a root of $P_A(x)$.

Exercise 16.1 Find the eigenvalues and eigenvectors of the following matrices

(a) $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **(c)** $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1/2 & -1/6 & 1/6 \\ 1/2 & -1/2 & -1/2 \\ -2/5 & 1/5 & 0 \end{bmatrix}$ **(e)** $A = \begin{bmatrix} -3 & -3 & 15 \\ -14 & 16 & 10 \\ -7 & -1 & 23 \end{bmatrix}$

Exercise 16.2 Let A be a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Express the coefficients of the characteristic polynomial $P_A(x)$ in terms of $\lambda_1, \lambda_2, \lambda_3$.

Exercise 16.3 A 3×3 migration matrix A has all entries between 0 and 1, and the sum of each column is 1. Thus A looks like

$$A = \begin{bmatrix} 1-x-y & u & v \\ x & 1-u-z & w \\ y & z & 1-v-w \end{bmatrix}.$$

Let P denote the initial population. Find the stable population distribution. (Hint: The stable population vector is fixed by A. Your answer will be a vector involving x, y, z, w, u, v.)