Chapter 18. A Tour of Four Dimensions

Four dimensional space $\mathbb{R}^4$ is the set of ordered quadruples of real numbers:

$$\mathbb{R}^4 = \{(x, y, z, w) : \ x, y, z, w \in \mathbb{R}\}.$$

Algebraically, there is no mystery about $\mathbb{R}^4$, or even higher dimensions. In fact everything in this chapter remains true, or generalizes in the obvious way, if 4 is replaced by n. We will stick to $n = 4$ for concreteness.

Some aspects of lower dimensions are hardly changed: The zero-vector is $\mathbf{0} = (0, 0, 0, 0)$ and the **standard basis** is the four vectors

$$\mathbf{e}_1 = (1, 0, 0, 0), \ \mathbf{e}_2 = (0, 1, 0, 0), \ \mathbf{e}_3 = (0, 0, 1, 0), \ \mathbf{e}_4 = (0, 0, 0, 1)$$

and every vector in $\mathbb{R}^4$ is uniquely a linear combination of the $\mathbf{e}_i$:

$$(x, y, z, w) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4.$$

A **line** in $\mathbb{R}^4$ is the set of scalar multiples of a given nonzero vector $\mathbf{v} \in \mathbb{R}^4$. Likewise, a **plane** in $\mathbb{R}^4$ is the set of linear combinations of two nonproportional vectors $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^4$. The $\{\textbf{dot product}\}$ between two vectors $\mathbf{v} = (x, y, z, w), \ \mathbf{v}' = (x', y', z', w')$ is defined by

$$\langle \mathbf{v}, \mathbf{v}' \rangle = xx' + yy' + zz' + ww',$$

the **length** of $\mathbf{v}$ is

$$|\mathbf{v}| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{x^2 + y^2 + z^2 + w^2}$$

and we have the geometric formula

$$\langle \mathbf{v}, \mathbf{v}' \rangle = |\mathbf{v}||\mathbf{v}'| \cos\theta$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{v}'$, measured in any plane containing them. The cross-product does not generalize in an obvious way; we'll get to that later. First, let's describe some objects which are new in four dimensions.

A **hyperplane** in $\mathbb{R}^4$ is the set of vectors $\mathbf{u} = (x, y, z, w)$ satisfying an equation of the form

$$ax + by + cz + dw = 0,$$

where a, b, c, d are fixed real numbers, not all zero. Alternatively, a hyperplane is the set of all linear combinations

$$x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3$$

of three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ not lying in a common plane. Since it takes three numbers (x, y, z) to specify a point, a hyperplane looks like $\mathbb{R}^3$ and we call it **three dimensional**. Among the infinitely many hyperplanes, there are four **coordinate hyperplanes**:

$$\begin{array}{rll} (x=0) & \text{denotes the hyperplane} & \{(0,y,z,w):\ y,z,w\in\mathbb{R}\} \\ (y=0) & \text{denotes the hyperplane} & \{(x,0,z,w):\ x,z,w\in\mathbb{R}\} \\ (z=0) & \text{denotes the hyperplane} & \{(x,y,0,w):\ x,y,w\in\mathbb{R}\} \\ (w=0) & \text{denotes the hyperplane} & \{(x,y,z,0):\ x,y,z\ \in\mathbb{R}\}. \end{array}$$

Note that the (general) hyperplane

$$ax + by + cz + dw = 0$$

consists of the vectors orthogonal to the **normal vector** $\mathbf{n} = (a, b, c, d)$, which must be nonzero. The intersection of two hyperplanes

$$\begin{aligned} ax + by + cz + dw &= 0 \\ a'x + b'y + c'z + d'w &= 0 \end{aligned}$$

with non-proportional normal vectors $\mathbf{n} = (a, b, c, d)$ and $\mathbf{n}' = (a', b', c', d')$ is a plane. For example,

$$(x = 0) \cap (w = 0) = \{(0, y, z, 0) :\ y, z \in \mathbb{R}\}$$

is the yz-plane, spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$.

Now consider the intersection of two planes (as opposed to hyperplanes). In 3d two planes usually intersect in a line, but in 4d things are different:

Usually, two planes in four dimensions intersect in the single point **0**.

For example, a point on the yz-plane has $x = w = 0$, while a point on the xw-plane has $y = z = 0$, so the only point on both planes is $(0, 0, 0, 0)$.

To explain what "usually" means in 4d, we will descibe the intersection in terms of a 4×4 matrix. Think of each plane itself as the intersection of two hyperplanes, so that we have four hyperplanes

$$a_{i1}x + a_{i2}y + a_{i3}z + a_{i4}w = 0, \qquad i = 1, 2, 3, 4. \tag{1}$$

The intersection of the two planes is the intersection of these four hyperplanes; A point (x, y, z, w) lives on the intersection exactly when it satisfies all four equations (1). In other words, the point must satisfy the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Writing this matrix equation concisely as

$$A\mathbf{v} = \mathbf{0},$$

we see that our intersection of two planes, or four hyperplanes, is the kernel of the matrix A:

$$\ker A = \{\mathbf{v} \in \mathbb{R}^4 : \ A\mathbf{v} = \mathbf{0}\},$$

where the rows of A are the coefficients in the four hyperplanes. The intersection is a single point exactly when $\ker A$ is trivial, in which case we write $\ker A = \mathbf{0}$.

For a general 4×4 matrix A, there are five possible dimensions of $\ker A$:

$\ker A = \mathbf{0}$	$\dim \ker A = 0$
$\ker A =$ a line	$\dim \ker A = 1$
$\ker A =$ a plane	$\dim \ker A = 2$
$\ker A =$ a hyperplane	$\dim \ker A = 3.$
$\ker A = \mathbb{R}^4$	$\dim \ker A = 4.$

As before, the determinant tells us if $\ker A$ is trivial or not.

$$\boxed{\ker A = \mathbf{0} \qquad \text{exactly when} \qquad \det(A) \neq 0.} \tag{2}$$

The determinant is defined recursively as before. If we let A_{ij} be the 3×3 matrix obtained from A by deleting row i and column j then for any row i we have

$$\det(A) = \sum_{j=1}^{4} (-1)^{i+j} a_{ij} \det(A_{ij})$$

and for any column j we have

$$\det(A) = \sum_{i=1}^{4} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Properties 1-6 of chapter 14 continue to hold. No matter how you expand $\det(A)$, you get the same sum of 24 terms

$$\begin{aligned}\det(A) &= \sum_{\sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} a_{4\sigma(4)} \\ &= a_{11}a_{22}a_{33}a_{44} - a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} \pm \cdots ,\end{aligned} \tag{3}$$

where the sum is over all $4! = 24$ permutations σ of the numbers $1, 2, 3, 4$ and $\operatorname{sgn}(\sigma) = \pm 1$ is itself the determinant

$$\operatorname{sgn}(\sigma) = \det(A_\sigma)$$

of the permutation matrix A_σ given by $A\mathbf{e}_i = \mathbf{e}_{\sigma(i)}$.

Our only purpose for writing out a few terms of the expansion (3) is to make it clear that $\det(A)$ is a polynomial expression in the entries of A. This clarifies our earlier assertion that "usually" two planes (or four hyperplanes) intersect in a single point. Indeed, we have seen that this intersection is the kernel of the matrix A whose rows are the hyperplanes being intersected, and $\ker A = \mathbf{0}$ exactly when the polynomial $\det$ is nonzero at A, which is what usually happens, for a random matrix A.

When $\ker A$ is larger than $\mathbf{0}$, it has a nonzero dimension which can be computed using determinants of submatrices of A. To explain this we first need some definitions. Fix $1 \leq k \leq 4$, and choose a pair of k-element subsets

$$I = \{i_1, \dots, i_k\}, \qquad J = \{j_1, \dots, j_k\}$$

of $\{1, 2, 3, 4\}$. Let a_{IJ} be the determinant of the $k \times k$ matrix whose entry in row p column q is $a_{i_p j_q}$. We call a_{IJ} a **k-minor** of A. For example, a 1-minor is just an entry a_{ij} in A and $\det(A)$ itself is the unique 4-minor of A.

The **rank** of A, denoted $\operatorname{rank}(A)$, is a number in $\{0, 1, 2, 3, 4\}$ defined as follows. If A is the zero matrix then $\operatorname{rank}(A) = 0$. Otherwise, $\operatorname{rank}(A)$ is the largest $k \geq 1$ for which some k-minor is nonzero. If all k-minors are zero, then all $(k+1)$-minors are zero automatically, so $\operatorname{rank}(A) < k$ and you need not bother with higher minors.

In general, the dimension of the kernel of A is given by the formula

$$\boxed{\dim \ker A = 4 - \operatorname{rank}(A)} \tag{4}$$

The recipe for computing $\ker A$ when $\det(A) = 0$ is as follows. There are four possibilities:

- A is the zero matrix. Then $\ker A = \mathbb{R}^4$.
- A is not the zero matrix, but all rows are proportional to each other. Then $\operatorname{rank}(A) = 1$ and $\ker A$ is the hyperplane with equation $ax+by+cz+dw = 0$, where (a, b, c, d) is any nonzero vector proportional to all the rows of A.
- Not all rows of A are proportional to each other, but all 3-minors of A are zero. Then $\operatorname{rank}(A) = 2$ and $\ker A$ is the plane given by the intersection of any two non-proportional row-hyperplanes.
- Some 3-minor a_{IJ} is nonzero. Then $\operatorname{rank}(A) = 3$ and $\ker A$ is a line. To find this line, let $i \in \{1, 2, 3, 4\}$ be the index not in I. For $j = 1, 2, 3, 4$, let A_{ij} be the 3×3 matrix obtained by deleting row i and column j. Then $\ker A$ is the line through the vector

$$(\det(A_{i1}),\ -\det(A_{i2}),\ \det(A_{i3}),\ -\det(A_{i4})), \tag{5}$$

which is nonzero because, if j is the number not in J, then $\det(A_{ij}) = a_{IJ} \neq 0$.

Example: The matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix}$$

has nonzero 2-minors, for example $\det \begin{bmatrix} 5 & 7 \\ 13 & 15 \end{bmatrix} = -16$. But all sixteen of the 3-minors are zero. For example,

$$\det \begin{bmatrix} 0 & 2 & 3 \\ 4 & 6 & 7 \\ 12 & 14 & 15 \end{bmatrix} = 0.$$

Therefore $\operatorname{rank}(A) = 2$ and $\dim \ker(A) = 4 - 2 = 2$. In other words, the four row hyperplanes of A intersect not in the usual point, but in a plane. Moreover, the plane $\ker A$ is the intersection of any two row hyperplanes of A.

To minimize the calculation of determinants, you can sometimes use **row reduction** to simplify a matrix before trying to find its rank and kernel, as follows. Suppose $\mathbf{u}$ and $\mathbf{v}$ are two rows of A. Take any nonzero scalars a, b and replace row $\mathbf{u}$ by $a\mathbf{u} + b\mathbf{v}$, giving a new matrix A' which differs from A only in the row that was $\mathbf{u}$ and is now $a\mathbf{u} + b\mathbf{v}$. Then we have

$$\operatorname{rank}(A) = \operatorname{rank}(A'), \qquad \ker(A) = \ker(A').$$

In the example above, we replace each of rows $2, 3, 4$ by itself minus the row immediately above, and get

$$A' = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}.$$

This shows that $\ker(A)$ is the intersection of the two hyperplanes

$$y + 2z + 3w = 0, \qquad x + y + z + w = 0.$$

The Four Dimensional Cross Product

As mentioned at the beginning of this chapter, the four dimensional analogue of the cross product is not so obvious. Take *three* vectors $\mathbf{u},\ \mathbf{v},\ \mathbf{w}$ in $\mathbb{R}^4$ and make the 3×4 matrix

$$C = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

whose rows are $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Let c_j be the determinant of the 3×3 matrix obtained by deleting column j from C. Then the **cross-product** of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the vector

$$\boxed{\mathbf{u} \times \mathbf{v} \times \mathbf{w} = (c_1\ ,\ -c_2\ ,\ c_3\ ,\ -c_4) \in \mathbb{R}^4.} \tag{6}$$

The notation is a bit subtle. In four dimensions $\mathbf{u} \times \mathbf{v}$ has not been defined. (It can be, but not in an introductory course, so we shall not consider it.) Likewise in three dimensions, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ are defined, but they are not equal, so $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ is not defined in three dimensions. The notation $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ applies only to four dimensions.

All but one of the properties of the 3d cross-product $\mathbf{u} \times \mathbf{v}$ have analogues for the 4d cross-product $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$. For example,

1. The vector $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and changes by a sign if you switch any two of the vectors.

2. The length of $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ is the volume of the parallelopiped
$$P = \{r\mathbf{u} + s\mathbf{v} + t\mathbf{w} :\ r, s, t \in [0, 1]\} \subset \mathbb{R}^4.$$

3. We have $\mathbf{u} \times \mathbf{v} \times \mathbf{w} \neq \mathbf{0}$ exactly when the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ span a hyperplane, whose coefficients are then those of $\mathbf{u}\times\mathbf{v}\times\mathbf{w}$. If $\mathbf{u}\times\mathbf{v}\times\mathbf{w} = \mathbf{0}$ then $\mathbf{u}, \mathbf{v}, \mathbf{w}$ live in a plane and do not span a hyperplane.

4. If A is a 4×4 matrix with $\operatorname{row}_i(A) = \mathbf{u}_i$, then
$$\det(A) = \langle \mathbf{u}_1, \mathbf{u}_2 \times \mathbf{u}_3 \times \mathbf{u}_4 \rangle.$$

5. The inverse of A exists exactly when $\det(A) \neq 0$, in which case
$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} \mathbf{u}_2 \times \mathbf{u}_3 \times \mathbf{u}_4 & -\mathbf{u}_1 \times \mathbf{u}_3 \times \mathbf{u}_4 & \mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_4 & -\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3 \end{bmatrix}.$$
(The minus signs were present in the 3×3 case as well, but we avoided them by re-ordering the factors in the middle cross-product. Here, it is probably easier for calculations if you keep the vectors in order and accept the signs.)

6. If A is a 4×4 matrix of rank 3 and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are rows of A in which some 3-minor is nonzero, then $\ker A$ is the line
$$\ker A = \mathbb{R}(\mathbf{u} \times \mathbf{v} \times \mathbf{w}).$$

7. The analogue of the 3d formula $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ is a bit more complicated. You are not required to know this, but I write it here, just to complete the story. Each of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is the edge of a tetrahedron. Take the edge $\mathbf{u}$. Let α and β be the angles made by $\mathbf{u}$ with $\mathbf{v}$ and $\mathbf{w}$, respectively. Let γ be the angle of the two faces of the tetrahedron meeting at the edge $\mathbf{u}$. Then
$$|\mathbf{u} \times \mathbf{v} \times \mathbf{w}| = |\mathbf{u}||\mathbf{v}||\mathbf{w}| \sin\alpha \sin\beta \sin\gamma.$$
If you start with a different edge then the angles will be different, but the product of sines will be the same.

Exercise 18.1 *Find the ranks and the kernels of the following matrices.*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Exercise 18.2 *Find the inverses of the following matrices.*

$$A = \begin{bmatrix} 0 & 0 & 0 & d \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}, \quad abcd \neq 0, \qquad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Exercise 18.3 *Use the cross product to find the equation of the hyperplane spanned by the vectors*

$$\mathbf{u} = (1,1,1,1), \quad \mathbf{v} = (1,2,1,2), \quad \mathbf{w} = (4,3,2,1).$$

Exercise 18.4 *Find a nonzero* 3*-minor and use the cross-product to find the kernel of the matrix*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix}.$$

Exercise 18.5 *Let* P *be the plane given as the intersection of two hyperplanes*

$$ax + by + cz + dw = 0, \qquad a'x + b'y + c'z + d'w = 0,$$

with non-proportional normal vectors $\mathbf{n} = (a,b,c,d)$ *and* $\mathbf{n}' = (a',b',c',d')$. *Let* N *be the plane spanned by the vectors* $\mathbf{n}$ *and* $\mathbf{n}'$. *What is the geometric relationship between the planes* P *and* N? *Justify your answer.* Hint: Dot product.

Exercise 18.6 *The characteristic polynomial $P_A(x) = \det(xI - A)$ has the same pattern as before: The coefficient of x^{4-k} in $P_A(x)$ is $(-1)^k$ times the sum of the diagonal k-minors of A. Compute $P_A(x)$ for the matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & d \\ 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \end{bmatrix}.$$

Exercise 18.7 *The eigenvalues of A are the roots λ of the characteristic polynomial $P_A(x)$ and the λ-eigenspace is $E(\lambda) = \ker(\lambda I - A)$. Find the eigenvalues of the matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

You will find that ± 1 are two of the eigenvalues. Compute $E(1)$ and $E(-1)$.