Chapter 21. Hooking Up in Hyperspace: A Love Story

Let S^n be a collection of points in $\mathbb{R}^{n+1}$ which are equidistant from a given point. We have

$$\begin{aligned} S^0 &= \text{two points} \\ S^1 &= \text{circle} \\ S^2 &= \text{sphere} \\ S^3 &= \text{hypersphere.} \end{aligned}$$

While originally defined in $\mathbb{R}^{n+1}$, an S^n can live in a higher dimensional space. For example, a circle can live in $\mathbb{R}^2,\ \mathbb{R}^3,\ \mathbb{R}^4,\ldots$ The superscript n denotes the intrinisic dimension of S^n, which is independent of the space in which the S^n lives. Dimension is how many numbers it takes to specify direction of motion. Motion on a circle is specified by a single number, the change in angle, so we call it S^1. Motion a sphere is specified by the change in longitude and latitude, so we call it S^2. Motion on two points is impossible, so we call it S^0.

As a way of understanding $\mathbb{R}^4$ or higher dimensional spaces, we'll look at how an S^n and an S^m (possibly with $n \neq m$) can be linked together in such a way that neither one is trapped inside the other, but you cannot pull them far apart without breaking or tearing one of them.

We'll start at the beginning, in $\mathbb{R}^1$, where we can only have a pair of S^0s. Denote one S^0 by $\bullet \quad \bullet$ and the other S^0 by $\circ \quad \circ$. They can be either linked:

$$\bullet \quad \circ \quad \bullet \quad \circ$$

or unlinked:

$$\bullet \quad \bullet \quad \circ \quad \circ\,.$$

Now, if we put this pair of S^0s in $\mathbb{R}^2$, the linkage disappears: we can move the middle two points $\circ \quad \bullet$ around each other and pull the two S^0s far apart.

In $\mathbb{R}^2$, however, an S^0 can be linked with an S^1, whereby one of the points of the S^0 (the prisoner) is inside the S^1 and the other (her boyfriend) is outside. This linkage again disappears in $\mathbb{R}^3$, where now two types of linkage are possible: An S^0 with an S^2 (same sad story) or a pair of interlocking S^1s (rings=happy ending). Summarizing the results so far, we have the following linkages:

$$S^0 + S^0 \subset \mathbb{R}^1$$

$$S^0 + S^1 \subset \mathbb{R}^2$$

$$S^0 + S^2 \subset \mathbb{R}^3, \qquad S^1 + S^1 \subset \mathbb{R}^3.$$

We are thus tempted to conjecture:

In $\mathbb{R}^{d+1}$ we have linkages $S^n + S^m$ exactly when $n + m = d$.

For this to be true in $\mathbb{R}^4$, four things must happen: the linkages S^0+S^2 and S^1+S^1 should disappear, and we should have new linkages $S^0 + S^3$ and $S^1 + S^2$.

This means the prisoner escapes her spherical cell to join her fiancé, only to find the wedding rings no longer linked. Then she is recaptured and placed in a hyperspherical cell while her faithful swain must exchange one of the rings for a sphere.

We can see exactly how all of this happens using the coordinates (x, y, z, w). Let's say the prisoner, originally in $\mathbb{R}^3$, is at $(0, 0, 0)$, inside the sphere with equation $x^2 + y^2 + z^2 = 1$. View this $\mathbb{R}^3$ as the hyperplane $(w = 0)$ in $\mathbb{R}^4$. Now the prisoner is at $(0, 0, 0, 0)$ and the sphere is

$$S^2 = \{(x, y, z, 0) : \; x^2 + y^2 + z^2 = 1\}.$$

She escapes from this S^2 via a path consisting of three line segments:

$$\begin{array}{lll} (0,0,0,t), & 0 \le t \le 1, & \text{then} \\ (0,0,s,1), & 0 \le s \le 2, & \text{then} \\ (0,0,2,1-t), & 0 \le t \le 1. & \end{array}$$

At no point does she touch the S^2, and her final point $(0, 0, 2, 0)$ is back in the original hyperplane $(w = 0)$, but now outside the S^2. According to her jailers, who cannot see beyond three dimensions, she simply disappeared (poof!) and reappeared outside (repoof!) high up on the z-axis, where the helicopter was waiting.

The rings become unlinked in a similar way. Let's say the rings, initially in $\mathbb{R}^3$, are

$$A = \{(x, y, 0) : \; x^2 + y^2 = 1\}, \qquad B = \{(0, y, z) : \; (y-1)^2 + z^2 = 1\}.$$

Again viewing $\mathbb{R}^3$ as the hyperplane $(w = 0)$ in $\mathbb{R}^4$ these rings become

$$A = \{(x, y, 0, 0) : \; x^2 + y^2 = 1\}, \qquad B = \{(0, y, z, 0) : \; (y-1)^2 + z^2 = 1\}.$$

We move A along the same escape path used by our heroine. Thus, each point $(x, y, 0, 0)$ on A goes to

$$\begin{array}{lll} (x,y,0,t), & 0 \le t \le 1, & \text{then} \\ (x,y,s,1), & 0 \le s \le 2, & \text{then} \\ (x,y,2,1-t), & 0 \le t \le 1. & \end{array}$$

At no point do the rings touch, and at the end of the path, our escapee ring is

$$A' = \{(x, y, 2, 0) : \ x^2 + y^2 = 1\}$$

which is unlinked from B. In both stories, our escapee lived in the xy plane and the two dimensions in the z, w coordinates gave enough wiggling room to unlink.

Now for the new linkages. Our brave prisoner is at $(0, 0, 0, 0)$, inside the hypersphere

$$S^3 = \{(x, y, z, w) : \ x^2 + y^2 + z^2 + w^2 = 1\}$$

consisting of all points in $\mathbb{R}^4$ at unit distance from her. During any escape our heroine's distance from $(0, 0, 0, 0)$ must increase continuously from zero to a large number, passing 1 along the way. But that means she hits the hypersphere, which is forbidden. This explains the linkage $S^0 + S^3 \subset \mathbb{R}^4$.

The most interesting part of the story is how the sphere and the circle are linked in $\mathbb{R}^4$ when they are placed as follows:

$$S^2 = \{(x, y, z, 0) : \ x^2+y^2+z^2 = 1\}, \qquad S^1 = \{(0, 0, z, w) : \ (z-1)^2+w^2 = 1\}.$$

The intersection of this S^1 with the hyperplane $(w = 0)$ is an S^0, consisting of two points

$$\circ = (0, 0, 0), \qquad \bullet = (0, 0, 2)$$

one inside the S^2, the other outside. Since we are in four dimensions, you may think we could wiggle around like before to get both of these points outside the S^2. But now we'll have to wiggle the whole circle containing these two points.

Imagine the circle in a plane, initially the zw plane as above, that moves in $\mathbb{R}^4$ as we wiggle. Stand on this plane, as though surfing, so that everything else seems to be moving except you and the plane. The intersection of the plane with the hyperplane $(w = 0)$ is now a line moving on your surfboard. Initially it was the line $\{(0, 0, z, 0) : \ z \in \mathbb{R}\}$, cutting your circle in the two points $\circ, \bullet$ with $z = 0, 2$. As the line moves, these points move, and if we attempt to unlink, the line will eventually move off the circle. As this happens, the two points coalesce when the line is tangent to the circle, and then disappear as soon as the line misses the circle.

What does this look like in the hyperplane $(w = 0)$? Remember we started with $\circ$ inside the S^2, and $\bullet$ outside. As the line moves toward the edge of the circle, the points get closer together. But one is inside and one is outside, so they must coalesce on the S^2. This means our S^1 has touched the S^2, which is forbidden.

Notice that the linkage $S^1 + S^2 \subset \mathbb{R}^4$ boils down to the linkage $S^0 + S^2 \subset \mathbb{R}^3$. Thus, the strength of the wedding ring/ball linkage is founded upon our heroine's previous incarceration in three dimensions. However, she is imprisoned anew, so the story continues...

Exercise 19.1 *Show that the sphere $S^2 = \{(x, y, z, 0) : \; x^2 + y^2 + z^2 = 1\}$ and the line $L = \mathbb{R}\mathbf{e}_4$ are linked in $\mathbb{R}^4$, in the sense that they do not touch, but cannot be pulled far apart.*
Hint: Consider the intersection of L with the plane $(w = 0)$.