Exam 2 Study Problems: Chapters 13-17

1. *Find a vector lying on the intersection of the two planes*

$$x + y + z = 0, \qquad x + 2y + 3z = 0.$$

Solution: The line of intersection contains the cross-product of the normal vectors $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 2, 3)$, so a desired vector is

$$\mathbf{u} \times \mathbf{v} = (1, -2, 1).$$

Any nonzero scalar multiple of this is correct.

2. *Find the cosine of the angle θ between the vectors*

$$\mathbf{u} = (1, 1, 1), \qquad \mathbf{v} = (1, 2, 3).$$

What is the area of the parallelogram spanned by $\mathbf{u}$ *and* $\mathbf{v}$*?*

Solution:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{|\mathbf{u}||\mathbf{v}|} = \frac{6}{\sqrt{42}}.$$

The area is

$$|\mathbf{u} \times \mathbf{v}| = |(1, -2, 1)| = \sqrt{6}.$$

3. *Find the volume of the parallelopiped spanned by*

$$\mathbf{u} = (1, 1, 0), \qquad \mathbf{v} = (1, 0, 1), \qquad \mathbf{w} = (0, 1, 1).$$

Solution: *The volume is*

$$\left| \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right| = 2.$$

4. *Find the kernels of the following matrices.*

$$\begin{bmatrix} 4 & 5 & 2 \\ 3 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & -1 & -1 \\ 2 & 7 & 3 \\ -1 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution:

line: $\mathbb{R}(1,-2,3)$, line: $\mathbb{R}(1,-5,11)$, plane: $x+y+z=0$, plane: $y+z=0$.

5. *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}$. *Find* A^{-1} *and the characteristic polynomial of* A.

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -3 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}, \quad P_A(x) = x^3 - 10x^2 + 7x - 1$$

6. *Find the eigenvalues and eigenspaces of the matrices*

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & 10 & -10 \\ 10 & 22 & -20 \\ 15 & 30 & -28 \end{bmatrix}.$$

Solution:
A: Evals $1, 2, 3$. $E(1) = \mathbb{R}(6,-5,1)$, $E(2) = \mathbb{R}(3,-4,1)$, $E(3) = \mathbb{R}(2,-3,1)$.
B: Evals $2, 2, -3$. $E(2)$=plane $x + 2y - 2z = 0$, $E(-3) = \mathbb{R}(1,2,3)$.

7. *Find a* 3×3 *matrix whose eigenvalues are* $1, -1, 2$*, with corresponding eigenvectors* $(1,2,3)$, $(1,3,5)$, $(1,3,6)$.

Solution: Let

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}.$$

Then B^{-1} was computed in problem 2. The desired matrix is

$$A = B \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} B^{-1} = \begin{bmatrix} 8 & -8 & 3 \\ 21 & -23 & 9 \\ 36 & -42 & 17 \end{bmatrix}.$$

8. *Solve the system of equations*

$$\begin{aligned} 2x + y + z &= 5 \\ x + y + z &= 6 \\ x + y + 2z &= 7. \end{aligned}$$

Hint: To solve $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, compute A^{-1}; the solution is $\mathbf{x} = A^{-1}\mathbf{y}$.

Solution: The inverse of the coefficient matrix is

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 1 \end{bmatrix}$$

so the solution is $x = -1,\ y = 6,\ z = 1$.

9. *We know that matrices*

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

are rotations, by angles of $\pi/2$, and $2\pi/3$ respectively. Find the axis and angle of rotation of AB and BA.

Solution: AB is rotation by π about the vector $(0, 1, 1)$, while BA is rotation by π about the vector $(1, 0, 1)$.

10. Find the matrix that rotates about the axis $\mathbb{R}(1, 0, 1)$ by an angle of $\pi/4$ as $(1, 0, 1)$ points at you.

Solution: The unit vector is $\frac{1}{\sqrt{2}}(1, 0, 1)$. Using the formula

$$A = I + \sin\theta U + (1 - \cos\theta)U^2,$$

we get

$$\begin{aligned} A &= I + \frac{1}{2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{\sqrt{2}-1}{2\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{2}+1 & -\sqrt{2} & \sqrt{2}-1 \\ \sqrt{2} & 2 & -\sqrt{2} \\ \sqrt{2}-1 & \sqrt{2} & \sqrt{2}+1 \end{bmatrix} \end{aligned}$$