Characters of $SL_2(q)$

Mark Reeder

December 23, 2008

1 The group SL(2) and some of its subgroups

Let $\mathfrak{f}$ be a finite field of cardinality $|\mathfrak{f}| = q$ and let $\mathfrak{f}^\times$ be the multiplicative group of nonzero elements in $\mathfrak{f}$.

$$G = SL_2(q) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \ a, b, c, d \in \mathfrak{f}, \ ad - bc = 1\}$$

is the group of 2×2 matrices over $\mathfrak{f}$ with determinant equal to 1. The group G acts transitively on the projective line $\mathbb{P}^1(\mathfrak{f})$ whose $q + 1$ points are the one-dimensional subspaces of $\mathfrak{f} \oplus \mathfrak{f}$. The kernel of this action is the center $Z = \{\pm I\}$ of G. Note that $|Z| = 2$ unless q is even, when $|Z| = 1$.

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the standard basis vectors of $\mathfrak{f} \oplus \mathfrak{f}$ and let $\ell_i = \mathfrak{f}e_i$ be the line through e_i. Given elements $x, y, \dots$ in a set with a G-action, let $\mathrm{Stab}_G(x, y, \dots) = \{g \in G : \ gx = x, gy = y, \dots\}$. Associated to the G-action on $\mathbb{P}^1(\mathfrak{f})$ we have subgroups

$$\begin{aligned} A &= \{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : \ a \in \mathfrak{f}^\times\} = \mathrm{Stab}_G(\ell_1, \ell_2), \\ B &= \{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : \ a \in \mathfrak{f}^\times, b \in \mathfrak{f}\} = \mathrm{Stab}_G(\ell_1), \\ U &= \{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : \ b \in \mathfrak{f}\} = \mathrm{Stab}_G(e_1). \end{aligned}$$

Since $|B| = q(q - 1)$ and $[G : B] = |\mathbb{P}^1(\mathfrak{f})| = q + 1$, we have $|G| = q(q^2 - 1)$.

The *Bruhat decomposition* is the expression of G as the disjoint union

$$G = B \cup BwB,$$

where $w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The $q + 1$ elements of $\{1\} \cup wU$ are coset representatives for $B\backslash G$ and

$$B^w \cap B = A.$$

2 Conjugacy classes

Let $\mathfrak{F}$ be a quadratic extension of $\mathfrak{f}$. The field $\mathfrak{F}$ has q^2 elements and is unique up to isomorphism. Let $x \mapsto \bar{x}$ be the nontrivial element in the Galois group $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$.

Any quadratic polynomial in $\mathfrak{f}[x]$ factors in $\mathfrak{F}[x]$. The eigenvalues of a matrix $g \in G$ are the roots of the quadratic polynomial $\det(xI - g) = x^2 - \mathrm{tr}(g)x + 1$, hence are of the form λ, λ^{-1}, where $\lambda \in \mathfrak{F}^\times$.

2.1 When q is odd

If q is odd, the squaring map on $\mathfrak{f}^\times$ has kernel $\{\pm 1\}$ of order two, so the squares in $\mathfrak{f}^\times$ have index two. We fix a non-square $\epsilon \in \mathfrak{f}^\times$, and choose an element $\sqrt{\epsilon} \in \mathfrak{F}$ whose square is ϵ. Then $\mathfrak{F} = \mathfrak{f}(\sqrt{\epsilon})$. For the eigenvalues λ, λ^{-1} of $g \in G$ we have three possibilities: They are both equal to ± 1, they are distinct in $\mathfrak{f}^\times$, or they are distinct in $\mathfrak{F}^\times - \mathfrak{f}^\times$.

2.1.1 $\lambda = \lambda^{-1} \in \{\pm 1\}$

Conjugating g so that e_1 is a λ-eigenvector, and further conjugating by elements of A we see that g is conjugate to one of

$$\pm I, \qquad \pm u, \qquad \pm u',$$

where

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad u' = \begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix}.$$

The elements $\pm u,\ \pm u'$ all have centralizer ZU, of order $2q$. Hence the four conjugacy-classes represented by $\pm u,\ \pm u'$ each consist of $\frac{1}{2}(q^2 - 1)$ elements.

2.1.2 $\lambda \neq \lambda^{-1} \in \mathfrak{f}^\times$

Here g is conjugate to the diagonal matrix

$$a = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in A$$

and also to a^{-1}. The centralizer of a is A. Since $|A| = q - 1$, the conjugacy class of a has $q(q+1)$ elements. There are $\frac{1}{2}(q-3)$ classes of this form, one for each pair $\{\lambda, \lambda^{-1}\}$.

2.1.3 $\lambda \neq \lambda^{-1} \notin \mathfrak{f}^\times$

Here $\lambda^{-1} = \bar{\lambda}$, where $z \mapsto \bar{z}$ is the Galois involution on $\mathfrak{F}/\mathfrak{f}$. Thus, λ belongs to the subgroup

$$S := \{z \in \mathfrak{F}^\times : \ z\bar{z} = 1\} = \{a + b\sqrt{\epsilon} : \ a, b \in \mathfrak{f}, \ a^2 - \epsilon b^2 = 1\}.$$

The norm homomorphism $z \mapsto z\bar{z}$ maps $\mathfrak{F}^\times$ onto $\mathfrak{f}^\times$. It follows that

$$|S| = q + 1.$$

The field $\mathfrak{F}$ is a vector space over $\mathfrak{f}$, on which S acts by multiplication. With respect to the basis $\{1, \sqrt{\epsilon}\}$, we have

$$S = \{\begin{bmatrix} a & \epsilon b \\ b & a \end{bmatrix} : \ a, b \in \mathfrak{f}, \ a^2 - \epsilon b^2 = 1\}.$$

This subgroup is analogous to the rotation group

$$SO_2 = \{\begin{bmatrix} a & -b \\ b & a \end{bmatrix} : \ a, b \in \mathbb{R}, \ a^2 + b^2 = 1\} \subset SL_2(\mathbb{R}).$$

In $SL_2(\mathfrak{F})$, S is conjugate to the subgroup

$$\{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} : \ \lambda \in \mathfrak{F}, \ \lambda\bar{\lambda} = 1\}.$$

It follows that the centralizer of any element $s \in S - Z$ is just S. Hence there are $q(q-1)$ elements in the conjugacy-class of s and we have $\frac{1}{2}(q + 1 - 2) = \frac{1}{2}(q-1)$ classes of this form.

2.1.4 Checking the results

The number of classes and elements in each class adds up to

$$2 + 4 \cdot \tfrac{1}{2}(q^2 - 1) + \tfrac{1}{2}(q-3) \cdot q(q+1) + \tfrac{1}{2}(q-1) \cdot q(q-1) = q(q^2 - 1) = |G|.$$

2.2 When q is even

If q is even, the squaring map on $\mathfrak{f}^\times$ has trivial kernel, hence is an isomorphism and there are no non-squares. The trace $\mathfrak{F} \to \mathfrak{f}$ is also surjective, so $\mathfrak{F}$ is generated by an element $\delta \in \mathfrak{F}$ with $\delta + \bar{\delta} = 1$, having minimal polynomial $x^2 + x + \epsilon$, where $\epsilon = \delta\bar{\delta}$.

For the eigenvalues λ, λ^{-1} of $g \in G$ we again have three possibilities: They are both equal to 1, they are distinct in $\mathfrak{f}^\times$, or they are distinct in $\mathfrak{F}^\times - \mathfrak{f}^\times$.

2.2.1 $\lambda = \lambda^{-1} = 1$

Conjugating g so that e_1 is a λ-eigenvector, and further conjugating by elements of A we see that if $g \neq I$ then g is conjugate to the element

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

which has centralizer U of order q and the conjugacy-class of u consists of $q^2 - 1$ elements.

2.2.2 $\lambda \neq \lambda^{-1} \in \mathfrak{f}^\times$

Here g is conjugate to the diagonal matrix

$$a = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

and also to a^{-1}. The centralizer of a is A. Since $|A| = q - 1$, the conjugacy class of a has $q(q+1)$ elements. There are $\frac{1}{2}(q-2)$ classes of this form, one for each pair $\{\lambda, \lambda^{-1}\}$.

2.2.3 $\lambda \neq \lambda^{-1} \notin \mathfrak{f}^\times$

Here $\lambda^{-1} = \bar{\lambda}$, where $z \mapsto \bar{z}$ is the Galois involution on $\mathfrak{F}/\mathfrak{f}$. Thus, λ belongs to the subgroup

$$S := \{z \in \mathfrak{F}^\times \ : \ z\bar{z} = 1\},$$

which again has order $q+1$. To see S as a subgroup of G, recall that $\mathfrak{F} = \mathfrak{f}(\delta)$, where $\delta^2 + \delta + \epsilon = 0$. With respect to the basis $\{1, \delta\}$, we have

$$S = \{\begin{bmatrix} a & \epsilon b \\ b & a+b \end{bmatrix} \ : \ a, b \in \mathfrak{f}, \ a^2 + ab + \epsilon b^2 = 1\}.$$

Arguing as in the q odd case, the centralizer of any element $s \in S - \{I\}$ is just S. Hence there are $q(q-1)$ elements in the conjugacy-class of s and we have $\frac{1}{2}(q+1-1) = \frac{1}{2}q$ classes of this form.

2.2.4 Checking the results

The number of classes and elements in each class adds up to

$$1 + 1 \cdot (q^2 - 1) + \tfrac{1}{2}(q-2) \cdot q(q+1) + \tfrac{1}{2}q \cdot q(q-1) = q(q^2-1) = |G|.$$

3 Representation Theory

3.1 Induced representations

Let G be a finite group, let H be a subgroup of G, and let $\sigma : H \to \mathbb{C}^\times$ be a linear character of H. For $x \in G$ set $H^x = x^{-1}Hx$.

The induced representation $\operatorname{Ind}_H^G \sigma$ has vector space

$$\operatorname{Ind}_H^G \sigma = \{f : G \to U : \ f(hg) = \sigma(h)f(g), \quad \forall h \in H, \ g \in G\}$$

and G action given by $[g \cdot f](x) = f(xg)$ for $x, g \in G$. The character of $\operatorname{Ind}_H^G \sigma$ is given by the formula

$$\operatorname{tr}(g, \operatorname{Ind}_H^G \sigma) = \sum_{\substack{x \in H \backslash G \\ g \in H^x}} \sigma(xgx^{-1}). \tag{1}$$

In particular, $\operatorname{tr}(g, \operatorname{Ind}_H^G \sigma) = 0$ unless g is conjugate to an element of H.

If $\rho : G \to GL(V)$ is another representation of G then we have Frobenius reciprocity:

$$\langle \rho, \operatorname{Ind}_H^G \sigma \rangle_G = \langle \rho, \sigma \rangle_H$$

where $\langle \rho_1, \rho_2 \rangle_G = \dim \operatorname{Hom}_G(V_1, V_2)$ for any representations $\rho_i : G \to GL(V_i)$. We extend $\langle \ , \ \rangle_G$ linearly to virtual characters.

The restriction of $\operatorname{Ind}_H^G \sigma$ to another subgroup $K < G$ is given by Mackey's formula:

$$\operatorname{Ind}_H^G \sigma|_K = \bigoplus_{x \in H \backslash G / K} \operatorname{Ind}_{H^x \cap K}^K (\sigma^x|_{H^x \cap K}).$$

3.2 Irreducible principal series representations

Once again $G = SL_2(\mathfrak{f})$ and B is the subgroup of upper-triangular matrices in G. Let $\chi : \mathfrak{f}^\times \to \mathbb{C}^\times$ be a character of $\mathfrak{f}^\times$. Extend χ to a character σ_χ of B by

$$\sigma_\chi \left(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right) = \chi(a)$$

and let

$$\rho_\chi = \operatorname{Ind}_B^G \sigma_\chi.$$

This is called a *principle series* representation. We have

$$\dim \rho_\chi = [G : B] = q + 1.$$

The character of ρ_χ can only be non-zero on the classes which meet B, which are represented by $\pm I, \pm u, \pm u'$ and $a(\lambda)$ for $\lambda \neq \pm 1$. (with the modifications above for even q). From formula (1), we have

$$\mathrm{tr}(\pm I, \rho_\chi) = \chi(\pm I)(q + 1). \tag{2}$$

Since $\pm u, \pm u'$ stabilize a unique line in $\mathfrak{f} \oplus \mathfrak{f}$, namely $\mathfrak{f}e_1$, these elements can only live in B^x if $x \in B$, thus (1) gives

$$\mathrm{tr}(\pm u, \rho_\chi) = \mathrm{tr}(\pm u', \rho_\chi) = \chi(\pm I). \tag{3}$$

The element $a(\lambda)$, with $\lambda \neq \pm 1$, stabilizes exactly two lines, namely $\mathfrak{f}e_1$ and $\mathfrak{f}e_2$. Hence $a(\lambda) \in B^x$ only if x preserves the set of these two lines. If $x \neq I$ then $x = vw$ for some $v \in U$, by the Bruhat decomposition. Of these elements, only w permutes the two lines fixed by $a(\lambda)$. Therefore (1) gives

$$\mathrm{tr}(a(\lambda), \rho_\chi) = \chi(\lambda) + \chi^w(\lambda) = \chi(\lambda) + \chi(\lambda^{-1}). \tag{4}$$

These character values hold for any χ. Let us now determine when ρ_χ is irreducible, and when two such representations are equivalent.

Proposition 3.1 *Let χ, χ' be characters of $\mathfrak{f}^\times$.*

1. *The principal series representation ρ_χ is irreducible if and only if $\chi^2 \neq 1$.*
2. *If $\chi^2 \neq 1 \neq (\chi')^2$, then $\rho_\chi = \rho_{\chi'}$ if and only if $\chi' \in \{\chi, \chi^{-1}\}$.*

Proof: By Frobenius reciprocity, Mackey's formula and the Bruhat decomposition we have

$$\langle \rho_{\chi'}, \rho_\chi \rangle_G = \langle \sigma_{\chi'}, \rho_\chi \rangle_B = \sum_{y \in B\backslash G/B} \langle \sigma_{\chi'}, \sigma_\chi^y \rangle_{B^y \cap B} = \langle \sigma_{\chi'}, \sigma_\chi \rangle_B + \langle \sigma_{\chi'}, \sigma_\chi^w \rangle_A = \langle \chi', \chi \rangle_{\mathfrak{f}^\times} + \langle \chi', \chi^{-1} \rangle_{\mathfrak{f}^\times}.$$

Taking $\chi' = \chi$, we see that

$$\langle \rho_\chi, \rho_\chi \rangle_G = \begin{cases} 1 & \text{if} \quad \chi^2 \neq 1 \\ 2 & \text{if} \quad \chi^2 = 1. \end{cases}$$

These calculations prove both assertions. ■

3.3 Reducible principal series representations

These are the representations ρ_χ, where χ is a character of $\mathfrak{f}^\times$ satisfying $\chi^2 = 1$, where $\langle \rho_\chi, \rho_\chi \rangle_G = 2$, so ρ_χ has two inequivalent constituents.

First suppose $\chi = 1$ is the trivial character. By Frobenius reciprocity, the trivial representation 1_G of G appears in ρ_1. The other constituent of ρ_1 is called the *Steinberg representation* and is denoted St_G. Thus, we have

$$\rho_1 = 1_G + St_G.$$

Since ρ_1 is just the permutation representation of G on $\mathbb{P}^1(\mathfrak{f})$, and 1_G is the subspace of constant functions, the vector space of St_G is given by

$$St_G = \{f : \mathbb{P}^1(\mathfrak{f}) \to \mathbb{C} \quad \text{such that} \quad \sum_{\ell \in \mathbb{P}^1(\mathfrak{f})} f(\ell) = 0\}$$

and the character of St_G has the formula

$$\mathrm{tr}(g, St_G) = |\mathbb{P}^1(\mathfrak{f})^g| - 1,$$

where the right side is the number of lines fixed by g, less one.

Next suppose $\chi^2 = 1$ but χ is not the trivial character. Such a χ exists if and only if q is odd, in which case it is unique; we denote it by χ_0. If $\mathfrak{f}^\times = \langle \gamma \rangle$, then χ_0 is given by

$$\chi_0(\gamma^j) = (-1)^j.$$

We abbreviate: $\rho_0 = \rho_{\chi_0}$. This representation has two inequivalent constituents:

$$\rho_0 = \rho_0' \oplus \rho_0''.$$

We now show that ρ_0 extends to an irreducible representation of $\tilde{G} = GL_2(q)$. For the subgroup

$$\tilde{B} = \{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : \; ad \neq 0\} \subset \tilde{G},$$

we again have the Bruhat decomposition: $\tilde{G} = \tilde{B} \cup \tilde{B}w\tilde{B}$ and $\tilde{B}^w \cap \tilde{B}$ is the group $\tilde{A}$ of diagonal matrices in $\tilde{G}$. Let χ be any nontrivial character of $\mathfrak{f}^\times$ and define a character $\tilde{\sigma}_0$ of $\tilde{B}$ by

$$\tilde{\sigma}_0\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = \chi(ad)\chi_0(a)$$

and let

$$\tilde{\rho}_0 = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}\,\tilde{\sigma}_0.$$

Restricting functions to G gives an equivalence of G-representations

$$\tilde{\rho}_0|_G \simeq \rho_0.$$

Thus, we have extended ρ_0 to $\tilde{G}$. Repeating the computation of Prop. 3.1, we get

$$\langle \tilde{\rho}_0, \tilde{\rho}_0 \rangle_{\tilde{G}} = \langle \tilde{\sigma}_0, \tilde{\rho}_0 \rangle_{\tilde{B}} = \sum_{y \in \tilde{B} \backslash \tilde{G} / \tilde{B}} \langle \tilde{\sigma}_0, \tilde{\sigma}_0^y \rangle_{\tilde{B}^y \cap \tilde{B}} = \langle \tilde{\sigma}_0, \tilde{\sigma}_0 \rangle_{\tilde{B}} + \langle \tilde{\sigma}_0, \tilde{\sigma}_0^w \rangle_{\tilde{A}}.$$

But now

$$\tilde{\sigma}_0^w \left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) = \tilde{\sigma}_0 \left(\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} \right) = \chi(ad)\chi_0(d).$$

Thus, $\tilde{\sigma}_0^w \neq \tilde{\sigma}_0$. It follows that $\langle \tilde{\rho}_0, \tilde{\rho}_0 \rangle_{\tilde{G}} = 1$ so $\tilde{\rho}_0$ is irreducible for $\tilde{G}$, as claimed.

We get partial information about the two G-constituents ρ_0' and ρ_0'' from the following:

Lemma 3.2 *Suppose $\tilde{\rho}$ is an irreducible character of $\tilde{G}$ whose restriction to G is a sum of two inequivalent irreducible characters σ and τ. Then*

1. $\dim \sigma = \dim \tau = \frac{1}{2} \dim \tilde{\rho}$.
2. $\sigma(a) = \tau(a)$ *and* $\sigma(s) = \tau(s)$ *for all* $a \in A$ *and* $s \in S$. *Thus,* $\sigma|_A = \tau|_A$ *and* $\sigma|_S = \tau|_S$.
3. $\sigma(\pm u') = \tau(\pm u)$ *and* $\sigma(\pm u) = \tau(\pm u')$.

Proof: Since G is normal in $\tilde{G}$ and $\tilde{\rho}$ is irreducible, the group $\tilde{G}$ permutes the irreducible G-constituents in the vector space of ρ. It follows that there is an element $x \in \tilde{G}$ such that $\sigma^{xgx^{-1}} = \tau(xgx^{-1}$ for all $g \in G$. Taking $g = I$ proves the first assertion.

We prove the other assertions by finding appropriate elements x. Since $\tilde{G}$ is permuting the two-element set $\{\sigma, \tau\}$, the subgroup $\tilde{G}_\sigma = \{y \in \tilde{G} : \ \sigma^y = \sigma\}$ has index two in $\tilde{G}$. Since $\tilde{G}_\sigma$ contains both G and the center $\tilde{Z}$ of $\tilde{G}$ (which acts by scalars on all of ρ), it follows that $G\tilde{Z} \subset \tilde{G}_\sigma$. On the other hand, $G\tilde{Z} = \{y \in \tilde{G} : \ \det y \in \mathfrak{f}^{\times 2}\}$, which implies that $G\tilde{Z} = \tilde{G}_\sigma$. It follows that if $x \in \tilde{G}$ has $\det x \in \mathfrak{f}^\times - \mathfrak{f}^{\times 2}$, then $\sigma^x = \tau$. One such element is

$$\tilde{a} := \begin{bmatrix} \epsilon & 0 \\ 0 & 1 \end{bmatrix} \in \tilde{A}. \tag{5}$$

Since $\tilde{a}$ centralizes A and $\tilde{a} u \tilde{a}^{-1} = u'$, it follows that

$$\sigma|_A = \tau|_A, \qquad \mathrm{tr}(u, \rho_0'') = \mathrm{tr}(u', \rho_0'), \qquad \mathrm{tr}(u', \rho_0') = \mathrm{tr}(u', \rho_0''). \tag{6}$$

Next we find such an element in the group

$$\tilde{S} = \{\begin{bmatrix} a & \epsilon b \\ b & a \end{bmatrix} : a^2 - \epsilon b^2 \neq 0\} \simeq \mathfrak{F}^\times,$$

which centralizes S. Recall that the norm mapping $\mathfrak{F}^\times \longrightarrow \mathfrak{f}^\times$ is surjective. Hence there exists $a + b\sqrt{\epsilon} \in \mathfrak{F}^\times$ with $a^2 - \epsilon b^2 = -1$ and the element

$$\tilde{s} = \begin{bmatrix} a & \epsilon b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} 0 & \epsilon \\ 1 & 0 \end{bmatrix}$$

has $\det \tilde{s} = \epsilon$. This completes the proof. ■

3.4 Cuspidal representations

The principal series representations and their irreducible constituents all come from characters of A. The cuspidal representations come from the group S, but in a more complicated way. We will only find the characters of these representations, without giving their vector spaces. More precisely, we will prove:

Proposition 3.3 *Let χ, η be characters of A and S respectively, having the same restriction to $Z = A \cap S$. Assume that $\eta \neq \bar{\eta}$. Then as virtual characters, we have*

$$\mathrm{Ind}_A^G \chi - \mathrm{Ind}_S^G \eta = \rho_\chi + \pi_\eta,$$

where π_η is an irreducible character of G of dimension $q - 1$ which does not depend on χ. If $\xi \neq \bar{\xi}$ is another character of S agreeing with η on Z then $\pi_\eta = \pi_\xi$ if and only if $\eta \in \{\xi, \bar{\xi}\}$.

Thus, the two induced representations $\mathrm{Ind}_A^G \chi$ and $\mathrm{Ind}_S^G \eta$ are almost isomorphic; their difference contains the constituent ρ_χ of $\mathrm{Ind}_A^G \chi$ and just one other irreducible constituent, which is π_η.
Proof: Recall that $\rho_\chi = \mathrm{Ind}_B^G \chi$. Let us define π_η by the above formula:

$$\pi_\eta = \mathrm{Ind}_A^G \chi - \mathrm{Ind}_B^G \chi - \mathrm{Ind}_S^G \eta.$$

We shall compute the three induced characters on the right side. In each case we need only evaluate them on conjugacy-classes that meet the inducing subgroup. For this, we need to know how these subgroups intersect their conjugates.

Lemma 3.4 *Let H be one of the subgroups A or S.*

1. *If $h \in H - Z$ then $H = C_G(h)$. In particular, H is its own centralizer in G.*
2. *H has index two in its normalizer $N_G(H)$ and $N_G(H)/H$ acts by inversion on H.*

3. For $x \in G$ we have

$$H^x \cap H = \begin{cases} H & \textit{if} \quad x \in N_G(H) \\ Z & \textit{if} \quad x \notin N_G(H). \end{cases}$$

Proof: You can prove this for A by simple matrix calculations, or by thinking about fixed-points in $\mathbb{P}^1(\mathfrak{f})$. We now prove it for S.

An element $s \in S$ has eigenvalues $\lambda, \bar{\lambda} \in \mathfrak{F}^\times$ with $\lambda\bar{\lambda} = 1$. Hence there is $g \in SL_2(\mathfrak{F})$ such that

$$gsg^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}.$$

Since S has order $q + 1 \geq 3$ we can choose $s \in S - Z$, so that $\lambda \neq \bar{\lambda}$. Then the centralizer of gsg^{-1} in $SL_2(\mathfrak{F})$ is the group $A(\mathfrak{F})$ of diagonal matrices in $SL_2(\mathfrak{F})$.

Let C be the centralizer of S in $SL_2(\mathfrak{f})$. Since S is abelian, we have $S < C$. If $x \in C$ then $gxg^{-1} \in A(\mathfrak{F})$. Suppose $x \in C - Z$. If the eigenvalues of x were in $\mathfrak{f}$, then since s centralizes x, the eigenvalues of s would also be in $\mathfrak{f}$ by our result for A, which they are not, since $\lambda \neq \bar{\lambda}$. Hence the eigenvalues of x are of the form $\mu, \bar{\mu}$ where $\mu\bar{\mu} = 1$ and gxg^{-1} is the diagonal matrix

$$gxg^{-1} = \begin{bmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{bmatrix}.$$

There are $q + 1$ choices for μ. We have shown that $|C| = |gCg^{-1}| = q + 1 = |S|$. Hence $C = S$, proving 1.

The element s and its inverse s^{-1} are the only elements of s having eigenvalues $\lambda, \bar{\lambda}$. If $n \in N_G(S)$ and $nsn^{-1} = s$ then $n \in S$, by part 1. It follows that $[N_G(S) : S] \leq 2$. We will write down an element $v \in N_G(S) - S$.

If q is even, then v is the Galois action on S. Recall that $\mathfrak{F} = \mathfrak{f}(\delta)$, where δ is a root of $x^2 + x + \epsilon$. The other root is $\bar{\delta} = 1 + \delta$. So using the basis $\{1, \delta\}$ as before, we have

$$v = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

If q is odd, the Galois matrix using the basis $\{1, \sqrt{\epsilon}\}$ is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which has determinant -1. If we again choose $a + b\sqrt{\epsilon} \in \mathfrak{F}^\times$ with $a^2 - \epsilon b^2 = -1$, then the element

$$v = \begin{bmatrix} a & \epsilon b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & -\epsilon b \\ b & -a \end{bmatrix}$$

has determinant $= 1$ and acts on S by inversion.

To prove 3, we must show that if $S^x \cap S$ contains an element outside of Z then $S^x \subseteq S$. Let $t \in S^x$ and suppose $s \in S^x \cap S$ with $s \notin Z$. Then t centralizes s since both elements are in the abelian group S^x. By part 1, we have $t \in S$. ■

Lemma 3.5 *The subgroup B is its own normalizer. For $x \in G$ we have*

$$B^x \cap A = \begin{cases} A & \text{if} \quad x \in Bw \\ Z & \text{if} \quad x \notin B \cup Bw \end{cases} \qquad \textit{and} \qquad B^x \cap ZU = \begin{cases} ZU & \text{if} \quad x \in B \\ Z & \text{if} \quad x \notin B. \end{cases}$$

Proof: Since the only line fixed by all of B is ℓ_1 it follows that B is its own normalizer. To compute B^x for any $x \in G$, we may assume $x \in wU$, by the Bruhat decomposition. The intersections of B^x with A and ZU can now be computed easily. ■

Using these lemmas 3.4 and 3.5, the character formula for an induced representation gives us the following traces, for $a \in A - Z$ and $s \in S - Z$, $u \in ZU - Z$:

$$\begin{aligned} \operatorname{tr}(a, \operatorname{Ind}_A^G \chi) &= \chi(a) + \chi(a^{-1}), \\ \operatorname{tr}(a, \operatorname{Ind}_B^G \chi) &= \chi(a) + \chi(a^{-1}), \\ \operatorname{tr}(zu, \operatorname{Ind}_B^G \chi) &= \chi(z), \\ \operatorname{tr}(s, \operatorname{Ind}_S^G \eta) &= \eta(s) + \eta(s^{-1}). \end{aligned} \tag{7}$$

These are the non-obvious entries in the following tables:

For q odd, with $\chi(-I) = \eta(-I) = e \in \{\pm 1\}$:

Class representative:	I	$-I$	u, u'	$-u, -u'$	a	s
number of such classes:	1	1	2	2	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-1)$
|class| :	1	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$
$\operatorname{Ind}_A^G \chi$	$q(q+1)$	$eq(q+1)$	0	0	$\chi(a)+\chi(a^{-1})$	0
$\operatorname{Ind}_B^G \chi$	$q+1$	$e(q+1)$	1	e	$\chi(a)+\chi(a^{-1})$	0
$\operatorname{Ind}_S^G \eta$	$q(q-1)$	$eq(q-1)$	0	0	0	$\eta(s)+\eta(s^{-1})$
π_η	$q-1$	$e(q-1)$	-1	$-e$	0	$-\eta(s)-\eta(s^{-1})$

We see that the virtual character π_η does not depend on the choice of χ which agrees with η on Z. We now show that π_η is an irreducible character if $\eta^2 \neq 1$.

For two characters η, ξ of S, we compute:

$$\begin{aligned}
|G|\langle \pi_\eta, \pi_\xi \rangle_G &= 2(q-1)^2 + 2(q^2-1) + q(q-1) \cdot \tfrac{1}{2} \sum_{S-Z} [\eta(s) + \bar{\eta}(s)][\xi(s) + \bar{\xi}(s)] \\
&= 2(q-1)^2 + 2(q^2-1) + \tfrac{1}{2} q(q-1) \left[\sum_{S} [\eta(s) + \bar{\eta}(s)][\xi(s) + \bar{\xi}(s)] - 8 \right] \\
&= 2(q-1)^2 + 2(q^2-1) + \tfrac{1}{2} q(q-1) \left[(q+1)\langle \eta + \bar{\eta}, \xi + \bar{\xi} \rangle_S - 8 \right] \\
&= |G| \cdot \tfrac{1}{2} \langle \eta + \bar{\eta}, \xi + \bar{\xi} \rangle_S \\
&= |G| \cdot \langle \eta, \xi + \bar{\xi} \rangle_S .
\end{aligned}$$

It follows that

$$\langle \pi_\eta, \pi_\xi \rangle_G = \langle \eta, \xi + \bar{\xi} \rangle_S . \tag{8}$$

Taking $\xi = \eta$ and assuming that $\eta \neq \bar{\eta}$, we have

$$\langle \pi_\eta, \pi_\xi \rangle_G = 1.$$

Hence $\pm\pi_\eta$ is an irreducible character of G. But $\pi_\eta(1) = q - 1 > 0$, so in fact π_η is an irreducible character of G. This, along with the formula (8), proves Prop. 3.3 in the case of odd q.

For q even, the calculation is slightly different, but leads to the same result. The character values are:

Class representative:	I	u	a	s
number of such classes:	1	1	$\frac{1}{2}(q-2)$	$\frac{1}{2}q$
|class| :	1	q^2-1	$q(q+1)$	$q(q-1)$
$\mathrm{Ind}_A^G \chi$	$q(q+1)$	0	$\chi(a) + \chi(a^{-1})$	0
$\mathrm{Ind}_B^G \chi$	$q+1$	1	$\chi(a) + \chi(a^{-1})$	0
$\mathrm{Ind}_S^G \eta$	$q(q-1)$	0	0	$\eta(s) + \eta(s^{-1})$
π_η	$q-1$	-1	0	$-\eta(s) - \eta(s^{-1})$

and for two characters η, ξ of S, the calculation of $\langle \pi_\eta, \pi_\xi \rangle_G$ is :

$$\begin{aligned}
|G|\langle \pi_\eta, \pi_\xi \rangle_G &= (q-1)^2 + (q^2-1) + q(q-1) \cdot \tfrac{1}{2} \sum_{S-Z} [\eta(s) + \bar{\eta}(s)][\xi(s) + \bar{\xi}(s)] \\
&= (q-1)^2 + (q^2-1) + \tfrac{1}{2} q(q-1) \left[\sum_{S} [\eta(s) + \bar{\eta}(s)][\xi(s) + \bar{\xi}(s)] - 4 \right] \\
&= (q-1)^2 + (q^2-1) + \tfrac{1}{2} q(q-1) \left[(q+1)\langle \eta + \bar{\eta}, \xi + \bar{\xi} \rangle_S - 4 \right] \\
&= |G| \cdot \tfrac{1}{2} \langle \eta + \bar{\eta}, \xi + \bar{\xi} \rangle_S \\
&= |G| \cdot \langle \eta, \xi + \bar{\xi} \rangle_S,
\end{aligned}$$

giving the same formula $\langle \pi_\eta, \pi_\xi \rangle_G = \langle \eta, \xi + \bar{\xi} \rangle_S$, which proves Prop. 3.3 in the case where q is even. ■

3.5 Reducible π_η

These occur when $\eta = \bar{\eta}$, where we have $\langle \pi_\eta, \pi_\eta \rangle_G = 2$. Since π_η is only a virtual character, this inner product tells us only that there are distinct irreducible characters π'_η, π''_η of G and signs $a, b \in \{\pm 1\}$ such that

$$\pi_\eta = a\pi'_\eta + b\pi''_\eta.$$

Indeed, if $\eta = 1_S$ is the trivial character of S, then

$$\pi_{1_S} = St_G - 1_G.$$

To see this, just note that the character of π_{1_G} has values:

Class representative:	I	$-I$	u	$-u$	a	s
π_{1_S}	$q-1$	$q-1$	-1	-1	0	-2

and observe that $\pi_{1_S}(g) = |\mathbb{P}^1(\mathfrak{f})^g| - 2 = \mathrm{tr}(g, St_G) - 1$.

If $\eta = \eta_0$ is the unique character of S having order two (which only exists when q is odd), the character $\pi_0 := \pi_{\eta_0}$ does turn out to be a sum of two distinct irreducible characters:

$$\pi_0 = \pi'_0 + \pi''_0.$$

To see this we proceed as we did for ρ_0: construct a character

$$\tilde{\pi}_0 = \mathrm{Ind}_{\tilde{A}}^{\tilde{G}} - \mathrm{Ind}_{\tilde{B}}^{\tilde{G}} - \mathrm{Ind}_{\tilde{S}}^{\tilde{G}}$$

of $\tilde{G} = GL_2(\mathfrak{f})$ using the groups $\tilde{A}$, $\tilde{B}$, $\tilde{S}$ from before. A calculation very similar to that for π_η shows that $\tilde{\pi}_0$ is an irreducible character for $\tilde{G}$. We can again apply Lemma 3.2 to see that

$$\dim \tilde{\pi}_0' = \dim \tilde{\pi}_0'' = \tfrac{1}{2}(q-1)$$

and that

$$\pi_0'|_A = \pi_0''|_A, \qquad \pi_0'|_S = \pi_0''|_S, \qquad \operatorname{tr}(\pm u, \pi_0'') = \operatorname{tr}(\pm u', \pi_0'), \qquad \operatorname{tr}(\pm u', \pi_0'') = \operatorname{tr}(\pm u, \pi_0'). \tag{9}$$

3.6 The character table for q even

For even q the character table is very simple. On one hand, the center Z is trivial and there is only one class of elements of order two, represented by u. On the other hand, the groups A and S have odd order, so they do not have characters of order two, and the representations $\rho_0', \rho_0'', \pi_0', \pi_0''$ do not arise. Hence, we have computed all the irreducible characters as shown:

Class representative:	I	u	a	s
number of such classes:	1	1	$\frac{1}{2}(q-2)$	$\frac{1}{2}q$
|class| :	1	q^2-1	$q(q+1)$	$q(q-1)$
1_G	1	1	1	1
St_G	q	0	1	-1
ρ_χ	$q+1$	1	$\chi(a)+\chi(a^{-1})$	0
π_η	$q-1$	-1	0	$-\eta(s)-\eta(s^{-1})$

There are four rows and four columns in this table, indexed by the families of characters and conjugacy classes defined above. The number of representations in row i equals the number of conjugacy classes in column i.

3.7 The character table for q odd

We have not yet computed the characters of $\rho_0', \rho_0'', \pi_0', \pi_0''$, but we have partial information from equations (6) and (9). These equations compute the characters of $\rho_0', \rho_0'', \pi_0', \pi_0''$ on A and S and express the characters on $\pm u, \pm u'$ in terms of just two numbers:

$$x = \operatorname{tr}(u, \rho_0'), \qquad y = \operatorname{tr}(u, \pi_0').$$

We also put $e_\chi = \chi(-1)$, $e_\eta = \eta(-1)$,

$$e = \chi_0(-1) = -\eta_0(-1) = \left(\frac{-1}{q}\right) = \begin{cases} 1 & \text{if} \quad q \equiv 1 \mod 4 \\ -1 & \text{if} \quad q \equiv -1 \mod 4. \end{cases}$$

With this notation, our character table becomes

Class:	I	$-I$	u	u'	$-u$	$-u'$	a	s
number:	1	1	1	1	1	1	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-1)$
|class| :	1	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$
1_G	1	1	1	1	1	1	1	1
St_G	q	q	0	0	0	0	1	-1
ρ_0'	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)e$	x	$1-x$	xe	$(1-x)e$	$\chi_0(a)$	0
ρ_0''	$\frac{1}{2}(q+1)$	$\frac{1}{2}(q+1)e$	$1-x$	x	$(1-x)e$	xe	$\chi_0(a)$	0
π_0'	$\frac{1}{2}(q-1)$	$\frac{1}{2}(1-q)e$	y	$-y-1$	$-ye$	$(y+1)e$	0	$-\eta_0(s)$
π_0''	$\frac{1}{2}(q-1)$	$\frac{1}{2}(1-q)e$	$-y-1$	y	$(y+1)e$	$-ye$	0	$-\eta_0(s)$
ρ_χ	$q+1$	$(q+1)e_\chi$	1	1	e_χ	e_χ	$\chi(a)+\bar{\chi}(a)$	0
π_η	$q-1$	$(q-1)e_\eta$	-1	-1	$-e_\eta$	$-e_\eta$	0	$-\eta(s)-\bar{\eta}(s)$

In this 8×8 table the number of irreducible characters in row i equals the number of conjugacy classes in column i (this number =1 for the first six rows and columns).

It remains to compute the values of x and y, which will determine the 4×4 square in middle of the table. For x, we compute:

$$0 = |G| \cdot \langle \rho_0', \rho_0'' \rangle = \tfrac{1}{2}(q+1)^2 + (q^2-1)[\bar{x}(1-x) + x(1-\bar{x})] + q(q+1) \cdot \tfrac{1}{2}(q-3),$$

which leads to the equation

$$4x\bar{x} - 2(x+\bar{x}) + 1 - q = 0. \tag{10}$$

Note that $\bar{x} = \operatorname{tr}(u^{-1}, \rho_0')$. If $e = +1$ then u^{-1} is conjugate to u, so $\bar{x} = x$. Equation (10) becomes a quadratic polynomial $4x^2 - 4x + 1 - q = 0$, with solutions $x = \frac{1}{2}[1 \pm \sqrt{q}]$. If $e = -1$ and u^{-1} is conjugate to u', so $\bar{x} = 1 - x$. Equation (10) becomes a quadratic polynomial $4x^2 - 4x + 1 + q = 0$, with solutions $x = \frac{1}{2}[1 \pm \sqrt{-q}]$. Thus, for all q we have

$$x = \tfrac{1}{2}[1 \pm \sqrt{eq}].$$

A similar argument works for y. From $\langle \pi_0', \pi_0'' \rangle_G = 0$ we get

$$4y\bar{y} + 2(y + \bar{y}) + 1 - q = 0. \tag{11}$$

If $e = +1$ then $\bar{y} = y$ and we get

$$y = \tfrac{1}{2}[-1 \pm \sqrt{q}].$$

If $e = -1$ then $\bar{y} = -y - 1$ and we get

$$y = \tfrac{1}{2}[-1 \pm \sqrt{-q}].$$

Hence for all q we have

$$y = \tfrac{1}{2}[-1 \pm \sqrt{eq}].$$

If we fix choices of $\sqrt{q}$ and $\sqrt{-q}$ (such choices cannot be made canonically) then the missing fragment of the character table becomes

	u	u'	$-u$	$-u'$
ρ_0'	$\frac{1}{2}[1 + \sqrt{eq}]$	$\frac{1}{2}[1 - \sqrt{eq}]$	$\frac{1}{2}[1 + \sqrt{eq}]e$	$\frac{1}{2}[1 - \sqrt{eq}]e$
ρ_0''	$\frac{1}{2}[1 - \sqrt{eq}]$	$\frac{1}{2}[1 + \sqrt{eq}]$	$\frac{1}{2}[1 - \sqrt{eq}]e$	$\frac{1}{2}[1 + \sqrt{eq}]e$
π_0'	$\frac{1}{2}[-1 + \sqrt{eq}]$	$\frac{1}{2}[-1 - \sqrt{eq}]$	$\frac{1}{2}[1 - \sqrt{eq}]e$	$\frac{1}{2}[1 + \sqrt{eq}]e$
π_0''	$\frac{1}{2}[-1 - \sqrt{eq}]$	$\frac{1}{2}[-1 + \sqrt{eq}]$	$\frac{1}{2}[1 + \sqrt{eq}]e$	$\frac{1}{2}[1 - \sqrt{eq}]e$

It is perhaps nicer to look at the linear combinations of the characters $\rho_0', \rho_0'', \pi_0', \pi_0''$, which contain the same information. Thus, we have the complete table for the following virtual characters:

	I	$-I$	u	u'	$-u$	$-u'$	a	s
$\frac{1}{2}[\rho_0' + \rho_0'' + \pi_0' + \pi_0'']$	q	e	0	0	e	e	$\chi_0(a)$	$-\eta_0(s)$
$\frac{1}{2}[\rho_0' + \rho_0'' - \pi_0' - \pi_0'']$	1	eq	1	1	0	0	$\chi_0(a)$	$\eta_0(s)$
$\frac{1}{2}[\rho_0' - \rho_0'' + \pi_0' - \pi_0'']$	0	0	0	0	$e\sqrt{eq}$	$-e\sqrt{eq}$	0	0
$\frac{1}{2}[\rho_0' - \rho_0'' - \pi_0' + \pi_0'']$	0	0	$\sqrt{eq}$	$-\sqrt{eq}$	0	0	0	0

3.8 Example: $q = 5$

Here $e = 1$ and G is the binary icosahedral group $SL_2(5)$. The subgroup A has unique inverse pairs of elements $\{a, a^{-1}\}$ and characters $\{\chi, \chi^{-1}\}$ of order 4. The subgroup S has unique inverse pairs $\{s, s^{-1}\}$ and $\{\eta, \eta^{-1}\}$ of order six, whose squares are inverse pairs of order three. Explicitly, we can take

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad u' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad s = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad s^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Let $\tau = \frac{1}{2}[1 + \sqrt{5}]$ be the golden ratio, and let $\tau' = \frac{1}{2}[1 - \sqrt{5}]$ be its algebraic conjugate. With this notation, the character table of $SL_2(5)$ is:

Class:	I	$-I$	u	u'	$-u$	$-u'$	a	s	s^2
$\lvert$class$\rvert$:	1	1	12	12	12	12	30	20	20
order :	1	2	5	5	10	10	4	6	3
1_G	1	1	1	1	1	1	1	1	1
St_G	5	5	0	0	0	0	1	-1	-1
ρ_0'	3	3	τ	τ'	τ	τ'	-1	0	0
ρ_0''	3	3	τ'	τ	τ'	τ	-1	0	0
π_0'	2	-2	$-\tau'$	$-\tau$	τ'	τ	0	1	-1
π_0''	2	-2	$-\tau$	$-\tau'$	τ	τ'	0	1	-1
ρ_χ	6	-6	1	1	-1	-1	0	0	0
π_η	4	-4	-1	-1	1	1	0	-1	1
π_{η^2}	4	4	-1	-1	-1	-1	0	1	1

The two-dimensional representations with characters π_0', π_0'' are the two embeddings of $SL_2(5)$ into S^3. Their projections to SO_3, as symmetries of the icosahedron, are the representations with characters ρ_0'' and ρ_0', respectively.

3.9 $PSL_2(q)$

Recall that $G = SL_2(q)$. We set $\bar{G} = G/Z = PSL_2(q)$, and let $\bar{g}$ be the projection to $\bar{G}$ of an element $g \in G$. If q is even then $\bar{G} = G$, so we assume q is odd, and we have

$$|\bar{G}| = \tfrac{1}{2}q(q^2 - 1).$$

The representations of $\bar{G}$ are those of G which are trivial on $-I$, so we can extract them from the table above. The conjugacy-classes in $\bar{G}$ are the projections of Z-orbits of conjugacy classes in G and the centralizer $C_{\bar{G}}(\bar{g})$ is the image of $C_G^{\pm}(g) = \{x \in SL_2(q) : \ xgx^{-1} = \pm g\}$. In fact, $C_G^{\pm}(g) = C_G(g)$ unless g has order four and g is conjugate to w. The element $\bar{w}$ belongs to the unique $\bar{G}$-conjugacy class of elements of order two. In this one case, $C_G^{\pm}(w)$ consists of two cosets of $C_G(w)$, so $|C_{\bar{G}}(\bar{w})| = |C_G(w)|$. For all other classes, we have $|C_{\bar{G}}(\bar{g})| = |C_G(g)/Z| = \frac{1}{2}|C_G(g)|$.

If $q \equiv 1 \mod 4$, then $\mathfrak{f}^{\times}$ contains an element i of order four and w is conjugate to the element $j = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in A$ and $\chi_0(-I) = 1$. This means ρ_0' and ρ_0'' are characters of $\bar{G}$. But S/Z has odd order, so $\eta_0(-I) = -1$ and π_0' and π_0'' are not characters of $\bar{G}$.

In the table below we denote class representatives by their pre-images in G.

Class:	I	u	u'	w	a	s
number:	1	1	1	1	$\frac{1}{4}(q-5)$	$\frac{1}{4}(q-1)$
|class| :	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}q(q+1)$	$q(q+1)$	$q(q-1)$
1_G	1	1	1	1	1	1
St_G	q	0	0	1	1	-1
ρ_0'	$\frac{1}{2}(q+1)$	$\frac{1}{2}[1+\sqrt{q}]$	$\frac{1}{2}[1-\sqrt{q}]$	$\chi_0(j)$	$\chi_0(a)$	0
ρ_0''	$\frac{1}{2}(q+1)$	$\frac{1}{2}[1-\sqrt{q}]$	$\frac{1}{2}[1+\sqrt{q}]$	$\chi_0(j)$	$\chi_0(a)$	0
ρ_χ	$q+1$	1	1	$2\chi(j)$	$\chi(a)+\bar{\chi}(a)$	0
π_η	$q-1$	-1	-1	0	0	$-\eta(s)-\bar{\eta}(s)$

If $q \equiv -1 \mod 4$, then we can choose $\epsilon = -1$. Then $w \in S$ and $\eta_0(-I) = 1$. This means π_0' and π_0'' are characters of $\bar{G}$. But A/Z has odd order, so $\chi_0(-I) = -1$ and ρ_0' and ρ_0'' are not characters of $\bar{G}$.

Class:	I	u	u'	w	a	s
number:	1	1	1	1	$\frac{1}{4}(q-3)$	$\frac{1}{4}(q-3)$
$\vert$class$\vert$:	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}q(q-1)$	$q(q+1)$	$q(q-1)$
1_G	1	1	1	1	1	1
St_G	q	0	0	1	1	-1
π_0'	$\frac{1}{2}(q-1)$	$\frac{1}{2}[-1+\sqrt{-q}]$	$\frac{1}{2}[-1-\sqrt{-q}]$	$-\eta_0(w)$	0	$-\eta_0(s)$
π_0''	$\frac{1}{2}(q-1)$	$\frac{1}{2}[-1-\sqrt{-q}]$	$\frac{1}{2}[-1+\sqrt{-q}]$	$-\eta_0(w)$	0	$-\eta_0(s)$
ρ_χ	$q+1$	1	1	0	$\chi(a)+\bar{\chi}(a)$	0
π_η	$q-1$	-1	-1	$-2\eta(w)$	0	$-\eta(s)-\bar{\eta}(s)$

For $q = 7$ we have $|\bar{G}| = 168$. We set $\alpha = \frac{1}{2}[-1+\sqrt{-7}]$ and the character table becomes

Class:	I	u	u'	w	a	s
number:	1	1	1	1	1	1
$\vert$class$\vert$:	1	24	24	21	56	42
1_G	1	1	1	1	1	1
St_G	7	0	0	-1	1	-1
π_0'	3	α	$\bar{\alpha}$	-1	0	1
π_0''	3	$\bar{\alpha}$	α	-1	0	1
ρ_χ	8	1	1	0	-1	0
π_η	6	-1	-1	2	0	0

Exercises

1. Determine the smallest dimension of a faithful representation of $G = SL_2(q)$, for each prime power q.

2. Show that $St_G|_U$ is the regular representation of U.

3. Show that the cuspidal representations π_η, π_0' and π_0'' are precisely the representations of G which do not contain the trivial representation of U.

4. Let $St_G^{(n)}$ be the Steinberg representation of $SL_2(q^n)$. Show that the restriction of $St_G^{(n)}$ to $G = SL_2(q)$ is the tensor product of St_G with itself n times:

$$St_G^{(n)}|_G = St_G^{\otimes n}.$$

Find the irreducible decomposition of this representation of G.

5. For $a, b \in \mathfrak{f}$ not both zero, let $[a, b]$ denote the one dimensional subspace of $\mathfrak{f} \oplus \mathfrak{f}$ spanned by $ae_1 + be_2$.

a) Show that if q is even then S acts simply-transitively on $\mathbb{P}^1(\mathfrak{f})$ and if q is odd then S has two orbits on $\mathbb{P}^1(\mathfrak{f})$, each of cardinality $\frac{1}{2}(q+1)$. What is the implication for the decomposition $\rho_\chi|_S$?

b) Assume q is odd. Show that the function $\nu : \mathbb{P}^1(\mathfrak{f}) \to \mathfrak{f}^\times/\mathfrak{f}^{\times 2}$ given by $\nu[a, b] = a^2 - \epsilon b^2 \mod \mathfrak{f}^{\times 2}$ is well-defined, and that its fibers are the S-orbits on $\mathbb{P}^1(\mathfrak{f})$.

6. Decompose $\mathrm{Ind}_A^G \chi$ and $\mathrm{Ind}_S^G \eta$ into irreducible representations of G. By Prop. 3.3, you need only do the calculations for one of these induced representations.

7. Compute $\pi_\eta|_S$. Observe the relation between η and this decomposition.

8. Prove that $SL_2(4) \simeq A_5$, and use the character table for general $SL_2(q)$ to work out the explicit character table of A_5. Compare this with the character table arising from the isomorphism $PSL_2(5) \simeq A_5$.

9. If N is a normal subgroup of a finite group H then some irreducible representation H is trivial on N. Use this to prove that $PSL_2(q)$ is simple if $q \geq 4$.

10. Use the character tables to show that $G = SL_2(q)$ has no subgroup H of index $< q$. (Hint: consider the permutation character on G/H.)