MT 453 Elements

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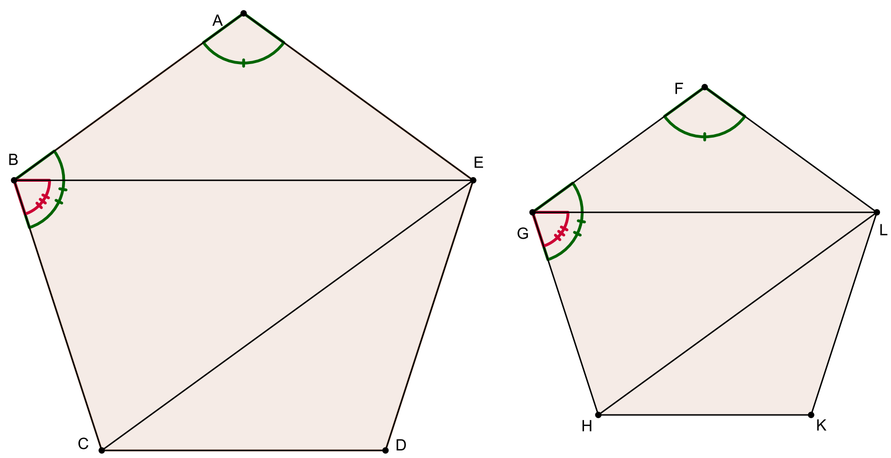
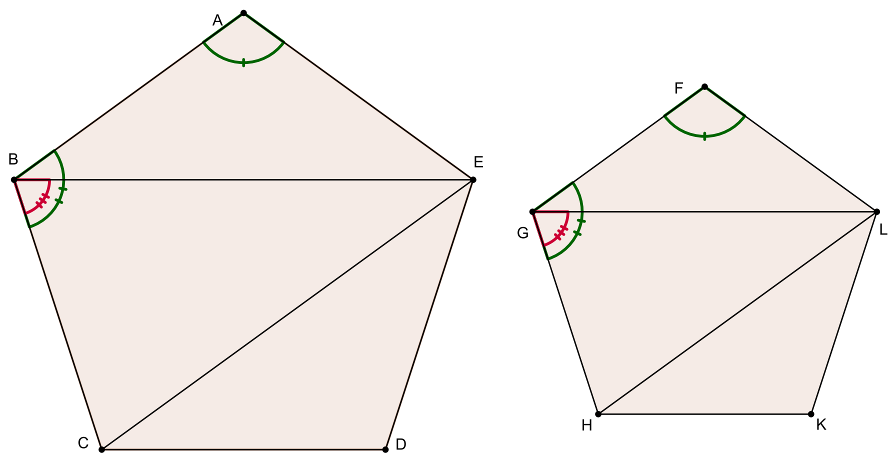
Proposition VI.20

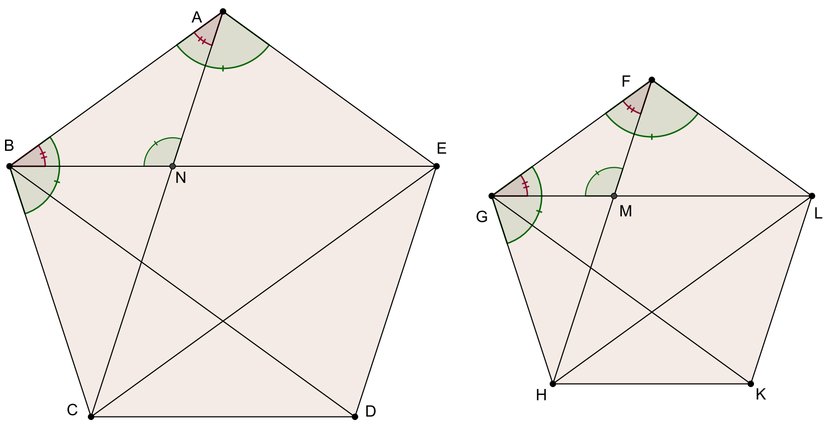
Similar polygons can be cut into similar triangles.



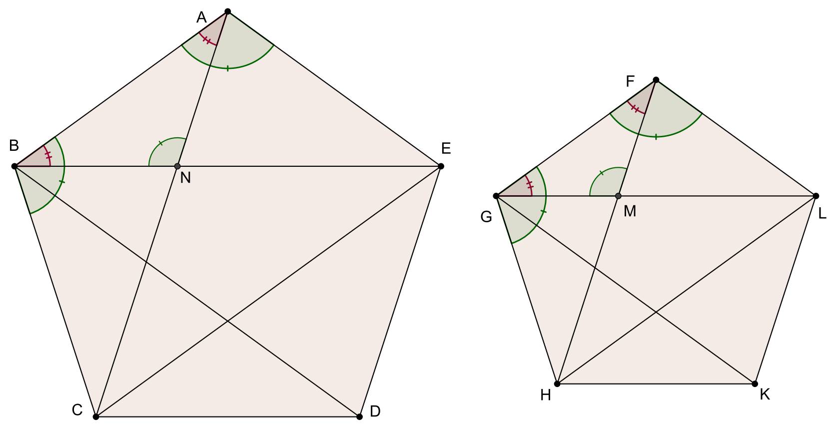
We are given Pentagon $ABCDE$ similar to Pentagon $FGHIJ$, with corresponding sides AB and FG.
Claim:
1. ΔABC is similar to ΔFGK, ΔBEC is similar to ΔGKL, andΔECD is simialr to ΔGHL.
2. $\frac{ABE}{FGL} = \frac{BEC}{GKL} = \frac{ECD}{GHL} = \frac{ABCDE}{FGHKL}$.
3. $\frac{ABCDE}{FGHKL} = \frac{AB^2}{FG^2}$.

1. $\angle BAE = \angle GFL$, so $\frac{AB}{AE} = \frac{GF}{FK}$. (Def. VI.I).
ΔBAE is equiangular to ΔGFL by SAS.
So ΔBAE is similar to ΔGFL (Prop. VI.4 and Def VI.6)
But $\angle ABC = \angle FGH$ because $ABCDE$ is simlar to $FGHKL$.
So the remaining $\angle EBC = \angle LGH$. (C.N.3)
Now $\frac{EB}{BA} = \frac{LG}{GF}$ sinceΔABE is similar to ΔGFL.
And $\frac{AB}{BC} = \frac{FG}{GH}$ since $ABCDE$ is similar to $FGHKL$.
So $\frac{EB}{BC} = \frac{LG}{GH}$ (ex aequali - V.22)
Thus ΔBEC is equiangular to ΔGLH (Prop. VI.6)
And ΔBEC is similar to ΔGHL. (Prop. VI.4 and Def VI.1)
Similarly, ΔECD is similar to ΔLHK.
So the first claim is proven.





2. Draw AC and FH.
Let the intersection ofAC and BE be called M, and let the intersection of FH and GL be called N.
We know $\angle ABC = \angle FGH$ and that $\frac{AB}{BC} = \frac{FG}{GH}$.
So ΔABC is equiangular to ΔFGH. (VI.6)
Thus $\angle BAC = \angle GFH$ implies $\angle BAM = \angle GFN$.
And $\angle ABE = \angle FGL$ implies $\angle ABM = \angle FGN$.
So the remaining $\angle AMB = \angle FNG$ (I.32).
Thus ΔAMB is equiangular to ΔFNG.
Similarly, ΔCMB is equiangular to ΔHNG.
So $\frac{AM}{BM} = \frac{FN}{G}N$.
And $\frac{BM}{MC} = \frac{GN}{NH}$.
Thus $\frac{AM}{MC} = \frac{FN}{NH}$. (ex aequali, V.22)
Now $\frac{AM}{MC} = \frac{\Delta AMB}{\Delta CEB}$. (VI.1)
And $\frac{AM}{MC} = \frac{\Delta AEM}{\Delta CEM}$.(VI.1)
So $\frac{AM}{MC} = \frac{\Delta AMB}{\Delta CEB} + \frac{\Delta AEM}{\Delta CEM} = \frac{\Delta ABE}{\Delta CBE}$. (V.12)
Similarly, $\frac{FN}{NH} = \frac{\Delta FGL}{\Delta GHL}$.
But since$\frac{AM}{MC} = \frac{FN}{NH}$, $\frac{\Delta ABE}{\Delta CBE} = \frac{\Delta FGL}{\Delta GHL}$. (V.11)
So $\frac{\Delta ABE}{\Delta FGL} = \frac{\Delta CBE}{\Delta GHL}$. (V.16)
Similarly, by drawing BD and GK, we can show that $\frac{\Delta CBE}{\Delta GHL} = \frac{\Delta CED}{\Delta HKL}$.
Thus $\frac{\Delta ABE}{\Delta FGL} = \frac{\Delta CBE}{\Delta GHL} = \frac{\Delta CED}{\Delta HKL}$.
And so $\frac{\Delta ABE}{\Delta FGL} = \frac{\Delta CBE}{\Delta GHL} = \frac{\Delta CED}{\Delta HKL} = \frac{\Delta ABE}{\Delta FGL} + \frac{\Delta CBE}{\Delta GHL} + \frac{\Delta CED}{\Delta HKL} = \frac{ABCDE}{FGHKL}$. (V.12)
So the second claim is proven.



3. ΔABE is similar to ΔFGL (as was shown above).
So $\frac{\Delta ABE}{\Delta FGL} = \frac{AB^2}{FG^2}$ (VI.19)
But $\frac{\Delta ABE}{\Delta FGL} = \frac{ABCDE}{FGHKL}$.
So $\frac{ABCDE}{FGHKL} = \frac{AB^2}{FG^2}$ (VI.11)
Thus the thrid claim is proven.
Q.E.D.

Porism:
For any similar rectilinear figures, they are to each other as the squares of the corresponding sides.