# MT 453 Elements Day 38

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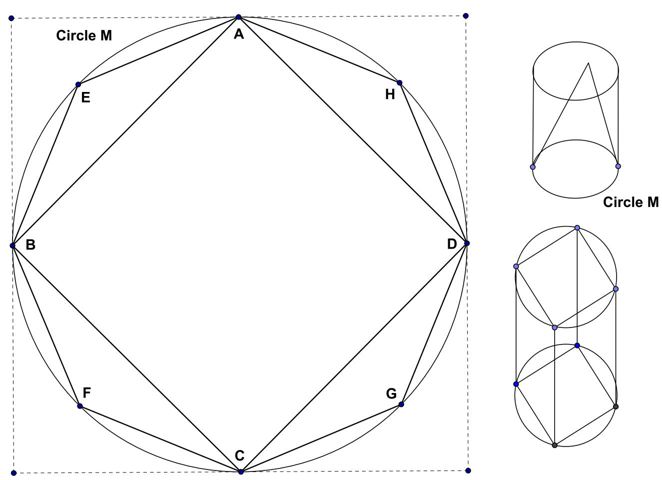
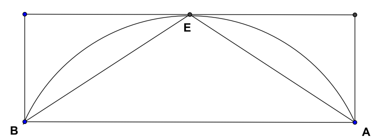
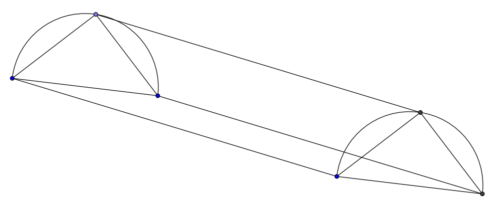
**Proposition XII.10**

*A cone with the same base and height as a cylinder is* $\frac{1}{3}$ *of the cylinder.*



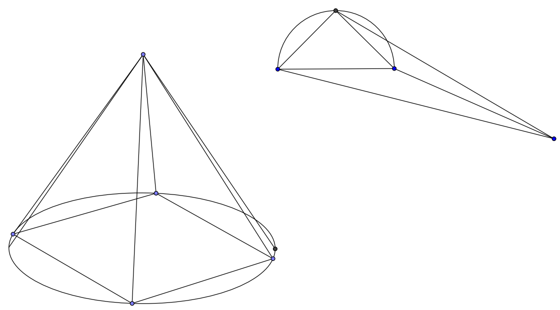

Given circle $M$ as the base of a cylinder and cone that have equal heights.
1. Let cylinder $M > 3$cone $M$
Inscribe square $ABCD$. (Prop. IV.6)
$ABCD = \frac{1}{2}$ outer square, and the outer square encloses circle $M$.
Therefore, $ABCD > \frac{1}{2}$ circle $M$.
Prism $ABCD = \frac{1}{2}$ prism outer square, where the prism outer square encloses cylinder $M$. (Prop. XI.32)
Therefore, prism $ABCD > \frac{1}{2}$ cylinder $M$.



Bisect the four circumferences and draw the four triangles. (Prop. X.1)
Then each $\triangle > \frac{1}{2}$ segment of circle $M$ that it lies on. (Prop. XII.2)
So, similarly, each of the four triangular prisms$>$ $\frac{1}{2}$the segment of cylinder $M$ that they lie on.
Remember, we assumed that cylinder$M > 3$cone$M$ or cylinder $M = 3$cone$M+\epsilon$.
Keep bisecting circumferences, etc., until what is left of cylinder $M < \epsilon$.
Let this happen at $E, F, G, H$.
Then prism $AH > 3$cone$M$ because what is left is less than $\epsilon$.
So prism $AH = 3$pyramd $AH$. (Porism of XII.7)
Then 3pyramid$AH > 3$cone$M$.
So pyramid $AH >$cone$M$ $\Rightarrow\Leftarrow$(since the cone encloses the pyramid.)
Then cylinder$M$ is not greater than 3cone$M$.

2. Let cylinder $M < 3$cone$M$ or $\frac{1}{3}$cylinder$M$ $<$cone$M$.
Look at pyramid $ABCD$ with same vertex as cone $M$.



As before, bisect the four circumferences and draw the four triangles. (Prop. X.1)
Each $\triangle > \frac{1}{2}$ segment on $M$.
Set up triangular pyramids on the segments of cone$M$.
Then each pyramid$> \frac{1}{2}$ segment cone $M$.
Remember, we assumed $\frac{1}{3}$cylinder$M <$cone$M$ or $\frac{1}{3}$cylinder$M + \epsilon =$cone$M$.
Keep bisecting, etc., until you are left with a portion of cone $M < \epsilon$.(Prop. X.1)
Let that happen at $E, F, G, H$.
Since what is left is less than $\epsilon$, pyramid $AH > \frac{1}{3}$ cylinder $M$.
But pyramid $AH = \frac{1}{3}$prism $AH$.
So 3pyramid $AH >$cylinder $M$.
This means prism $AH >$cylinder $M, \Rightarrow\Leftarrow$.

Therefore, cone $M = \frac{1}{3}$cylinder $M$.