MT 453 Elements Day 34
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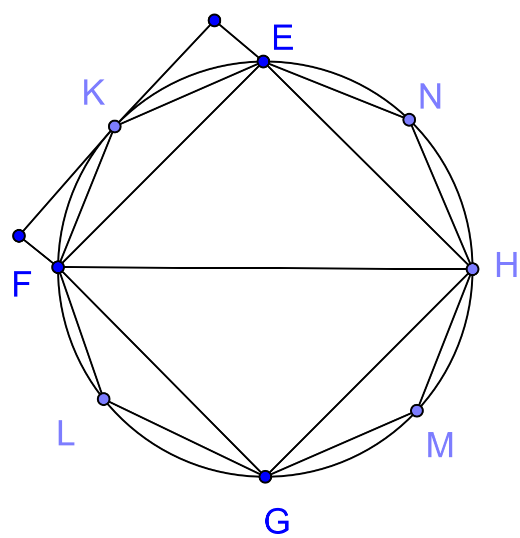
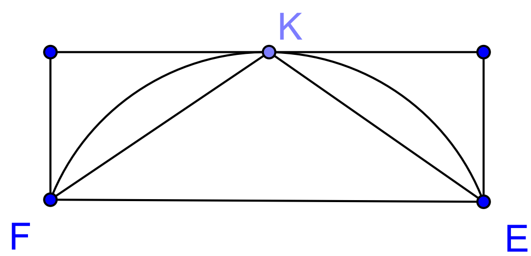
Proposition XII.2

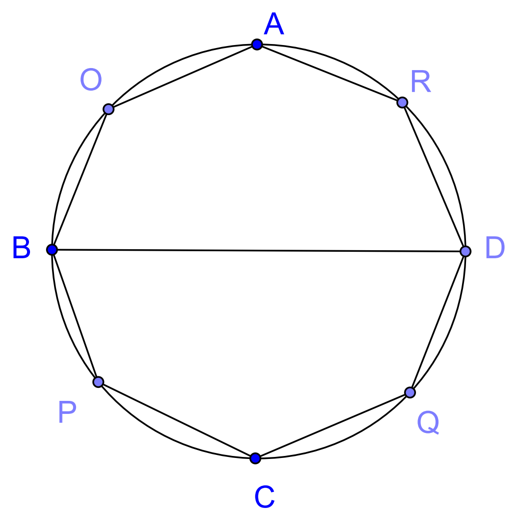
Circles are to one another as the squares on their diameters
Let $\odot ABCD$ and $\odot EFGH$ be circles with diameters BD and FG, respectively. Let S be a magnitude such that $BD^2 : FG^2 = \odot ABCD : S$. I claim that $S = \odot EFGH$.
Suppose $S < \odot EFGH$.
Inscribe a square $\Diamond EFGH$ in the circle $\odot EFGH$; then $\Diamond EFGH$ is greater than half of $\odot EFGH$.
Bisect the arcs EF, FH, HG, GE at K, L, M, N respectively.



Then $\triangle EKF$ is greater than half of arc EF, etc.



Continue bisecting until the remaining segments are less than $S - \odot EFGH$. [X.1]
This gives a polygon $[E]$ with diameter FG which is greater than S.
Inscribe a polygon $[A]$, with diameter BD, and similar to $[E]$, in $\odot ABCD$.



Then $BD^2 : FG^2 = [A] : [E]$, [XII.1]
so $\odot ABCD : S = [A] : [E]$, or $\odot ABCD : [A] = S : [E]$.
But $[E] > S$, so $\odot ABCD : [A]$, which is impossible.
Therefore we cannot have $BD^2 : FG^2 = \odot ABCD : S$ with $S < \odot EFGH$.
Suppose we have $BD^2 : FG^2 = \odot ABCD : S$ with $S > \odot EFGH$.
Then $FG^2 : BD^2 = S : \odot ABCD = \odot EFGH : T$, where $T < \odot ABCD$.
This too is impossible, by the the argument of the first part, with the circles interchanged.
Therefore we have $BD^2 : FG^2 = \odot ABCD : \odot EFGH$, so that the circles are as the squares on their diameters.
Q.E.D.