MT 453 Elements Day 35

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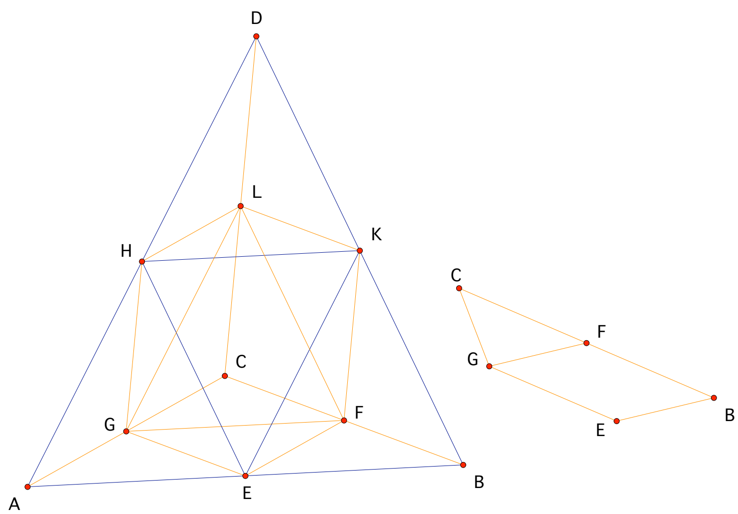
Proposition XII.3

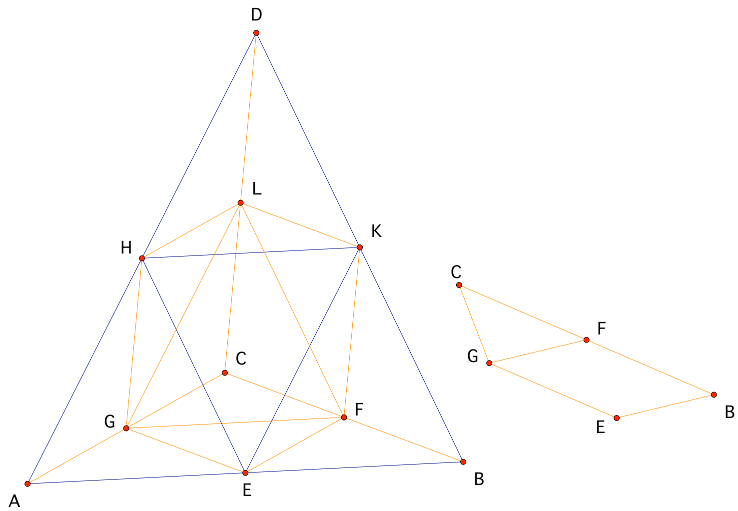
A triangular pyramid is made up of four solids:

- *Two congruent triangular pyramids similar to the large one*
- *Two equal triangular prisms, comprising more than $\frac{1}{2}$ the large pyramid.*



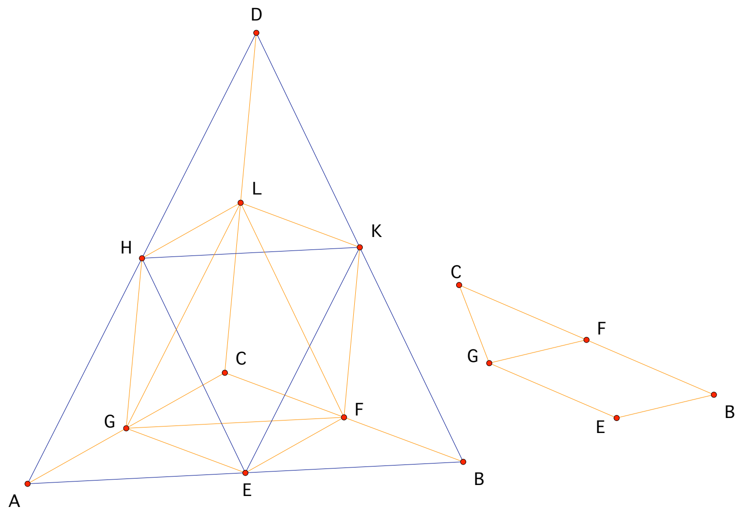
Proof:

Bisect every edge of the triangular pyramid.
There are four claims.



1. Claim: pyramid $HAEG \cong$ pyramid $DHKL$.

HK is parallel to AB. (Prop. VI.2)
HE is parallel to DB. (Prop. VI.2)
Therefore, $HKBE$ is a parallelogram.
$HK = EB$. (Prop. I.34)
So then $HK = AE$ because we bisected AB.
$HE = KB$. (Prop. I.34)
So then $HE = DK$ because we bisected DB.
But $AH = HD$ because we bisected AD.
So $\triangle AHE \cong \triangle HDK$. (SSS)
Similarly, $\triangle AGH \cong \triangle HLD$.
AE is parallel to HK.
AG is parallel to HL.
$\angle GAE = \angle LHK$. (Prop. XI.10)
$\triangle GAE \cong \triangle LHK$. (SAS)
Similarly, $\triangle HGE \cong \triangle DLK$. (SAS)
Therefore, pyramid $DHLK \cong$ pyramid $HAGE$. (Def. XI.10)



2. Claim: pyramid $DHLK$ is similar to pyramid $DACB$.

$\triangle DHK$ is similar to $\triangle DAB$.

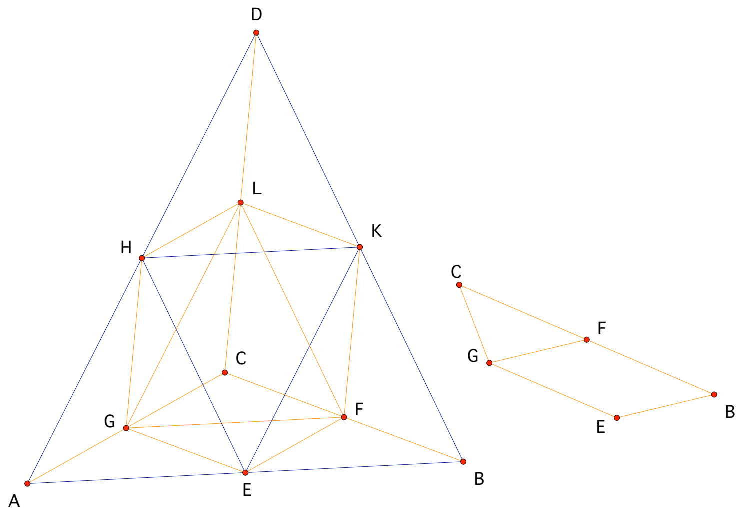
> They share $\angle ADB$. Also, $\angle DHK = \angle DAB$ because they are cut by parallel lines. Then $\angle HKD = \angle ABD$ because they are the remaining angles. The triangles are equiangular. The sides are in proportion because we bisected the sides of each. Therefore, the triangles are similar.

Similarly, $\triangle DLK$ is similar to $\triangle DCB$.
And also, $\triangle DLH$ is simmilar to $\triangle DCA$.
We also have $AB = 2HK$, $AC = 2HL$, and $CB = 2LK$, so $\triangle ACB$ is similar to $\triangle HLK$ in the same proportion.
So pyramid $DHLK$ is similar to pyramid $DACB$.

3. Claim: triangular prism $HGEKFB$ = triangular prism $HLKGCF$.

But $\lozenge EGFB$ and $\triangle FGC$ sit on equal bases ($BF = FC$) with same height (the perpendicular line from G to CF = the perpendicular line from E to FB since GE is parallel to CB).
So $\lozenge EGFB = 2\triangle FGC$. (Prop. I.41)
Done by Proposition XI.39.

> Given a prism P_1, with parallelogram base, and a prism P_2, with a triangle base, of the same height, if $\lozenge = 2\triangle$, then $P_1 = P_2$.



4. Claim: triangular prism $HGEKFB+$ triangular prism $HLKGCF > \frac{1}{2}$ pyramid $DACB$

Manifestly, $HGEKFB > KEBF$ (c.n.5)
Manifestly, $HGEKFB > HAEG$ (since $KEBF = HAEG$).
Similarly, $HLKGCF > LGCF = DHLK$.
So, together, $HGEKFB + HLKGCF > HAEG + DHLK$.

> The sum of these four solids is equal to the large pyramid $DACB$. But two of these solids, namely $HGEKFB$ and $HLKGCF$, are greater than the other two, namely $HAEG$ and $DHLK$. But these other two, $HAEG$ and $DHLK$ are equal to half of the large pyramid $DACB$. Therefore, we have that $HGEKFB + HLKGCF > \frac{1}{2}DACB$.

Q.E.D.

Comments:

1. In claim one, we prove the congruence of triangles AHE and HDK and of triangles AGH and HLD. These are coplanar, and so we can use the argument involving parallelograms to prove their congruence. However, the remaining two, triangles HGE and DLK are not coplanar, and therefore we use a different argument to prove their congruence.

2. At the end of our proof of claim one, we use Definition XI.10 to show that triangular pyramid $DHLK \cong$ triangular pyramid $HAGE$. This is similar to an SSS (side-side-side) argument for congruence, with the exception that it is for 3D solids now.

3. In claim two, we show that $\triangle ACB$ is similar to $\triangle HLK$ by showing their sides are in proportion. However, we could also use Proposition XI.10 to prove they are similar.

4. In claim three, we use Proposition I.41 to show that $\Diamond EGFB = 2\triangle FGC$. This proposition says that if parallelogram has the same base with a triangle and they are in the same parallels, then the parallelogram is double that of the triangle. We use this proposition, with the understanding that *same base* implies *equal bases*.

5. In the proof of claim four, we said that $HGEKFB > HAEG$ since $KEBF = HAEG$. We chose these two pyramids arbitrarily, but the equality is the same for $KEBF$ and any one of the three pyramids on the base of the large pyramid.