Linear Algebra Notes

Chapter 1

ARITHMETIC OF MATRICES

A 2×2 **matrix** is a square array of numbers

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

You can multiply a matrix by a number:

$$t \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ta & tb \\ tc & td \end{bmatrix}.$$

You can add, subtract and multiply two matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a-a' & b-b' \\ c-c' & d-d' \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{bmatrix}.$$

To remember the last formula, note that the entry in row i column j of the product is the dot product of row i in the left matrix times column j in the right matrix.

For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix},$$

while

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}.$$

Unlike numbers, matrix multiplication is in general *non-commutative* . We cannot always expect to have $AB = BA$ when A and B are matrices. However, it can happen that two particular matrices A and B may happen to commute. For example:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

The matrix analogue of the number 1 is the **Identity Matrix** :

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For any matrix A, we have

$$IA = AI = A.$$

The **inverse** of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The quantity $ad - bc$ is called the **Determinant** of A, and is denoted

$$\det A = ad - bc.$$

The inverse of A only exists if $\det A$ is nonzero. In that case we say A is **invertible,** and we have

$$AA^{-1} = A^{-1}A = I.$$

Exercise 1.1. *Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Compute $AB,\ BA$. *You should get different answers.*

Exercise 1.2. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *as in 1.1, and suppose* $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *where* $a+c = d,\ 2c = 3b$. *Show that* $AB = BA$.

Exercise 1.3. *Compute*

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

You should get the same answer both times.

Exercise 1.4. *Using the trig identities*

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi, \quad \sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta,$$

show that

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}.$$

Exercise 1.5. *The powers of a matrix are computed as* $A^2 = AA,\ A^3 = AAA,$ *etc. Let* $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. *Compute* $A^2,\ A^3,\ A^4, A^{100}$.

Exercise 1.6. *Find the inverses of*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}.$$

Exercise 1.7. *Let*

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Show that

$$A^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

Exercise 1.8. *There are only two numbers that are their own inverses, namely 1 and -1. Find five matrices that are their own inverses.*