Linear Algebra Notes

Chapter 10

COMPLEX EIGENVALUES

The last case to consider is when A has complex eigenvalues. These arise whenever oscillatory behavior is encountered.

Consider, for example, the following model of glucose-insulin interaction.

$$\begin{aligned} x_{n+1} &= 0.9x_n - 0.4y_n \\ y_{n+1} &= 0.1x_n + 0.9y_n \end{aligned}$$

Here

$$\begin{aligned} x_n = &\text{ amount of excess glucose at time } n, \\ y_n = &\text{ amount of excess insulin at time } n. \end{aligned}$$

"Excess" means the deviation from the fasting level. We start measuring at time $n = 0$, immediately after a meal. The initial condition is

$$x_0 = 100, \quad y_0 = 0$$

reflecting the influx of glucose, and the delay in the pancreatic response of insulin production. At subsequent times we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 90 \\ 10 \end{bmatrix}, \qquad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 77 \\ 18 \end{bmatrix}, \qquad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 62.1 \\ 23.9 \end{bmatrix} \quad \cdots$$
$$\begin{bmatrix} x_8 \\ y_8 \end{bmatrix} = \begin{bmatrix} -9.3 \\ 25.7 \end{bmatrix}, \qquad \begin{bmatrix} x_9 \\ y_9 \end{bmatrix} = \begin{bmatrix} -29 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} x_{10} \\ y_{10} \end{bmatrix} = \begin{bmatrix} 1.6 \\ -8.3 \end{bmatrix} \quad \cdots$$

Note that the levels dip below the baseline and climb back up. This is caused by oscillatory behavior. To analyze this, let

$$A = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

Then the glucose-insulin vector at time n is given by the formula

$$A^n \begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

To compute A^n, and determine the long-term behavior, we try to find the eigenvectors of A, as usual. We find the characteristic polynomial is

$$P_A(x) = x^2 - (1.8)x + 0.85,$$

so the eigenvalues are

$$\lambda, \mu = \frac{1}{2}[1.8 \pm i\sqrt{0.16}] = 0.9 \pm 0.2i$$

What to do with these complex eigenvalues? We will now study general matrices with complex eigenvalues, and then return to the example when we are ready to handle it.

Take a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with real entries, and suppose A has complex eigenvalues. This occurs when the characteristic polynomial $P_A(x)$ has complex roots. Recall that

$$P_A(x) = x^2 - (\mathrm{tr}\, A)x + \det A.$$

The roots of $P_A(x)$ are given by the quadratic formula, namely

$$\frac{1}{2}\left[\mathrm{tr}\, A \pm \sqrt{(\mathrm{tr}\, A)^2 - 4\det A}\,\right].$$

Let Δ be the *negative* of the quantity inside the square root:

$$\Delta = 4\det A - (\mathrm{tr}\, A)^2.$$

So A will have complex eigenvalues exactly when

$$\Delta > 0,$$

in which case the eigenvalues will be

$$\lambda = \frac{1}{2}[\mathrm{tr}\, A + i\sqrt{\Delta}], \qquad \bar{\lambda} = \frac{1}{2}[\mathrm{tr}\, A - i\sqrt{\Delta}].$$

The eigenvectors of A will have complex entries. For example, one λ-eigenvector is

$$\mathbf{u} = (b, \lambda - a) = (b, \frac{\mathrm{tr}\, A}{2} - a) + i(0, \frac{\sqrt{\Delta}}{2}) = (b, \frac{d-a}{2}) + i(0, \frac{\sqrt{\Delta}}{2}).$$

Let

$$B = \frac{1}{2}\begin{bmatrix} 2b & 0 \\ d-a & -\sqrt{\Delta} \end{bmatrix}. \tag{10a}$$

It turns out that

$$B^{-1}AB = \frac{1}{2}\begin{bmatrix} \mathrm{tr}\, A & -\sqrt{\Delta} \\ \sqrt{\Delta} & \mathrm{tr}\, A \end{bmatrix}. \tag{10b}$$

This can be seen just by multiplying the matrices, but that is an ugly computation (see exercise 10.8). Instead we proceed as we have done with real eigenvalues, but with now the complex numbers cause things to split in two parts. First of all, the complex eigenvalue λ is written above in the form

$$\lambda = \alpha + i\beta,$$

where $\alpha = (\operatorname{tr} A)/2$ and $\beta = \sqrt{\Delta}/2$ are real numbers, and the complex eigenvector $\mathbf{u}$ is written in the form

$$\mathbf{u} = \mathbf{u}_{re} + i\mathbf{u}_{im},$$

where $\mathbf{u}_{re}$ and $\mathbf{u}_{im}$ are real vectors. The matrix B is defined by

$$B\mathbf{e}_1 = \mathbf{u}_{re}, \qquad B\mathbf{e}_2 = -\mathbf{u}_{im}.$$

Now we compute $B^{-1}AB\mathbf{e}_1$ and $B^{-1}AB\mathbf{e}_2$ simultaneously, as follows.

$$\begin{aligned}
B^{-1}AB\mathbf{e}_1 - iB^{-1}AB\mathbf{e}_2 &= B^{-1}A\mathbf{u}_{re} + iB^{-1}A\mathbf{u}_{im} \\
&= B^{-1}A(\mathbf{u}_{re} + i\mathbf{u}_{im}) \\
&= B^{-1}A\mathbf{u} \\
&= B^{-1}\lambda\mathbf{u} \\
&= \lambda B^{-1}\mathbf{u} \\
&= \lambda B^{-1}(\mathbf{u}_{re} + i\mathbf{u}_{im}) \\
&= (\alpha + i\beta)(\mathbf{e}_1 - i\mathbf{e}_2) \\
&= (\alpha\mathbf{e}_1 + \beta\mathbf{e}_2) - i(-\beta\mathbf{e}_1 + \alpha\mathbf{e}_2).
\end{aligned}$$

Comparing real and imaginary parts, we get

$$B^{-1}AB\mathbf{e}_1 = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2, \qquad \text{and} \qquad B^{-1}AB\mathbf{e}_2 = -\beta\mathbf{e}_1 + \alpha\mathbf{e}_2,$$

which gives (10b) when we recall that $\alpha = (\operatorname{tr} A)/2$ and $\beta = \sqrt{\Delta}/2$.

The matrix in (10b) is called the **canonical form** of A. It is the closest we can get to a diagonal matrix. Notice that the first column of $B^{-1}AB$ consists of the real and imaginary parts of the eigenvalue λ, without the i, and the second column is obtained by switching the entries of the first column, and changing a sign. Thus,

$$B^{-1}AB = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

This looks like a rotation matrix, except that

$$\det(B^{-1}AB) = \det A$$

and this may not equal 1. However, since the quantity $\Delta = 4\det A - (\operatorname{tr} A)^2$ is positive, the quantity $\det A$ must at least be positive, so we can compute the real number

$$r = \sqrt{\det A}, \tag{10c}$$

and we can rewrite $B^{-1}AB$ as

$$B^{-1}AB = r\begin{bmatrix} \frac{\operatorname{tr} A}{2r} & -\frac{\sqrt{\Delta}}{2r} \\ \frac{\sqrt{\Delta}}{2r} & \frac{\operatorname{tr} A}{2r} \end{bmatrix}.$$

The matrix on the right, without the factor of r in front, is now a rotation matrix (it has determinant equal to one). That is,

$$\begin{bmatrix} \frac{\operatorname{tr} A}{2r} & -\frac{\sqrt{\Delta}}{2r} \\ \frac{\sqrt{\Delta}}{2r} & \frac{\operatorname{tr} A}{2r} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{10d}$$

for some angle θ. To find the angle θ of rotation, equate the ratios of the entries in the first column of both matrices in (10d):

$$(\frac{\sqrt{\Delta}}{2r})/(\frac{\operatorname{tr} A}{2r}) = (\sin\theta)/(\cos\theta),$$

that is,

$$\frac{\sqrt{\Delta}}{\operatorname{tr} A} = \tan\theta.$$

In other words,

$$\theta = \arctan\frac{\sqrt{\Delta}}{\operatorname{tr} A}. \tag{10e}$$

In summary, we have shown that if A is a matrix with complex eigenvalues, then there is an invertible matrix B such that

$$\begin{aligned} B^{-1}AB &= \frac{1}{2}\begin{bmatrix} \operatorname{tr} A & -\sqrt{\Delta} \\ \sqrt{\Delta} & \operatorname{tr} A \end{bmatrix} \\ &= r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \end{aligned} \tag{10f}$$

where

$$\Delta = 4\det A - (\operatorname{tr} A)^2, \qquad r = \sqrt{\det A}, \qquad \theta = \arctan\frac{\sqrt{\Delta}}{\operatorname{tr} A}.$$

The two matrices on the right side of (10f) are equally good ways of writing the canonical form of A. You can write these matrices down without knowing B. In any case, the matrix B is given in (10a).

If $r = 1$ then the matrix in (10f) is a rotation matrix, and the original matrix A is a "flattened rotation"-it rotates vectors around on ellipses. If $r \neq 1$ we could call the matrix in (10f) a **spiral matrix** , because for any vector $\mathbf{v}$, the vectors

$$r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\mathbf{v}, \quad r^2\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^2\mathbf{v}, \quad r^3\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^3\mathbf{v}, \ldots$$

move in a spiral around the origin. This means that the original matrix A moves in a "flattened spiral" where the flattening is done by B. If $r < 1$ the spiral moves in toward the origin (stable) and if $r > 1$ the spiral moves away from the origin (unstable).

The quantities r and θ are none other than the magnitude and angle of the complex eigenvalue λ. Indeed, you can plot any complex number $z = \alpha + i\beta$ in the xy plane, by thinking of it as a vector (α, β). The length of z is then $|z| = \sqrt{\alpha^2 + \beta^2}$ and its angle is the angle made with the x axis, namely $\arctan(\beta/\alpha)$. When you apply this to the complex number

$$\lambda = \frac{1}{2}[\operatorname{tr} A + i\sqrt{\Delta}]$$

you see that the length of λ is r and its angle is θ (picture drawn in class).

EXAMPLE: Before returning to the glucose/insulin example, lets look at simpler matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We have $r = \sqrt{\det A} = 1$, and $\Delta = 4 - 1 = 3$. The canonical form of A is

$$B^{-1}AB = \frac{1}{2}\begin{bmatrix} \operatorname{tr} A & -\sqrt{\Delta} \\ \sqrt{\Delta} & \operatorname{tr} A \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix},$$

and $\theta = \arctan(-\sqrt{3}) = \frac{2\pi}{3}$. This angle is one third of a circle, and corresponds to the fact that $A^3 = I$. Note that we could write down the canonical form immediately from equation (10f), without even finding the eigenvalues (which are $\lambda = \frac{1}{2}[-1 + i\sqrt{3}]$, $\bar{\lambda} = \frac{1}{2}[-1 - i\sqrt{3}]$). This example is continued in exercise 10.2.

GLUCOSE/INSULIN REVISITED: We have $r = \sqrt{0.85} = 0.92\cdots < 1$ so the vectors

$$A\begin{bmatrix} 100 \\ 0 \end{bmatrix}, \quad A^2\begin{bmatrix} 100 \\ 0 \end{bmatrix}, \quad A^3\begin{bmatrix} 100 \\ 0 \end{bmatrix}, \ldots$$

spiral in toward the origin. The origin represents the fasting level of glucose/insulin. This is what is supposed to happen in a healthy person-the levels oscillate until returning to the normal fasting values. The value of θ in this example is

$$\theta = \arctan(\frac{0.2}{0.9}) = 0.22\ldots.$$

This is the change in angle of the vector from one unit of time to the next.

Exercise 10.1. *For each of the following matrices A, find the canonical form, the eigenvalues, r and θ.*

(a) $A = \begin{bmatrix} 7 & -10 \\ 4 & -5 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 5 & -5 \\ 2 & -1 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

Exercise 10.2. *Find the eigenvalues of*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(Your answer will be in terms of θ.)

Exercise 10.3. *Take the matrix* $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ *from the example above. Plot the vectors* $\mathbf{e}_1, A\mathbf{e}_1, A^2\mathbf{e}_1$ *with a hollow dot (○). Plot* $\mathbf{e}_2, A\mathbf{e}_2, A^2\mathbf{e}_2$ *with a solid dot (●). Connect the dots with an ellipse (which will be slanted). Repeat this starting with* $t\mathbf{e}_1$, $t\mathbf{e}_2$ *for various scalars t. For each t you will get an ellipse, and these ellipses will concentrically fill up the plane. The matrix A is a flattened rotation around these ellipses.*

Exercise 10.4. *In this exercise we find the equations of the ellipses in exercise 10.2. The idea is very similar to exercise 6.3. Let A be as in exercise 10.3, and let $\lambda, \bar{\lambda}$ be the eigenvalues of A. Let*

$$f(x, y) = (y - \lambda x)(y - \bar{\lambda} x).$$

(a) *Show that $f(x, y) = x^2 + xy + y^2$.*
(b) *Given a point (x, y), define (x', y') by the equation*

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Show that

$$f(x', y') = f(x, y).$$

(c) *If you fix a number $k > 0$, the graph of $x^2 + xy + y^2 = k$ is an ellipse C_k. Show that if (x, y) is on the ellipse C_k, then (x', y') is on the same ellipse C_k.*

Exercise 10.5. *Suppose the glucose/insulin equations are*

$$\begin{array}{c} x_{n+1} = (1 - c)x_n - 0.4y_n \\ y_{n+1} = cx_n + 0.9y_n \end{array},$$

where $0 \le c \le 1$. Here c is the proportion of glucose absorbed, and cx_n is also the amount of insulin production stimulated by the presence of x_n units of glucose. For what values of c will the system oscillate? If it oscillates, is it stable or unstable?

Exercise 10.6. *If the spiral goes into the origin, then we say the origin is* **stable.** *If the spiral goes away from the origin, we say the origin is* **unstable.** *Look at each of the following matrices, and without writing any calculations at all, say if the origin is stable or unstable. (You are allowed to do one calculation (per matrix) in your head, but you are not allowed to write it down.)*

$$\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & -1 \\ 2 & -\frac{1}{3} \end{bmatrix}.$$

Exercise 10.7. *Can you find a matrix with all positive entries, and having complex eigenvalues?*

Exercise 10.8. *Prove equation (10b) by multiplying the matrices.*