

# Linear Algebra Notes

## Chapter 2

### SPECIAL TYPES OF MATRICES

A **scalar matrix** :

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

A **diagonal matrix** :

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

The product of two diagonal matrices is very simple:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix} = \begin{bmatrix} aa' + 0 & 0 + 0 \\ 0 + 0 & 0 + dd' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ 0 & dd' \end{bmatrix}.$$

Thus, diagonal matrices multiply like numbers. Note that if  $A$  and  $B$  are diagonal, then

$$AB = BA.$$

An **upper triangular matrix** :

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

In contrast to diagonal matrices, upper triangular matrices do not always commute with each other. For example:

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix},$$

while

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

A **symmetric matrix**:

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}.$$

You have, or will see these in multivariable calculus: The matrix of second partial derivatives of a function  $f(x, y)$  is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

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which is symmetric since  $f_{xy} = f_{yx}$ . Symmetric matrices will arise later in this course. The **transpose** of a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is the matrix  $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ . So a matrix  $A$  is symmetric exactly when  $A = A^T$ .

**A rotation matrix:**

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \text{where } a^2 + b^2 = 1.$$

Thus,  $(a, b)$  is a point on the unit circle. Let  $\theta$  be the angle of rotation from  $(1, 0)$  up to  $(a, b)$ . Then  $a = \cos \theta$ ,  $b = \sin \theta$ , so

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

This  $\theta$  is called the “angle of rotation”, for reasons that will become clear.

Examples of rotation matrices:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{here } \theta = \frac{\pi}{2}.$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{here } \theta = \frac{\pi}{4}.$$

**A reflection matrix:**

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad \text{where } a^2 + b^2 = 1.$$

Note that

$$A^2 = I.$$

In other words, reflection matrices are their own inverses.

Examples of reflection matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Rotations and reflections are exactly those matrices which satisfy  $A^{-1} = A^T$  (inverse equals transpose). The difference between them is that

$$\begin{aligned} \det A = +1 & \quad \text{if } A \text{ is a rotation,} \\ \det A = -1 & \quad \text{if } A \text{ is a reflection.} \end{aligned}$$

**Exercise 2.1.** *Show that scalar matrices commute with all other matrices.*

**Exercise 2.2.** Suppose  $a \neq d$ . Show that the only matrices commuting with  $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$  are diagonal matrices.

**Exercise 2.3.** Show that

$$(AB)^T = (B^T)(A^T).$$

Note the reversal of order in the products.

**Exercise 2.4.**

- a) Show that the product of two rotation matrices is again a rotation matrix.
- b) Show that the product of two reflection matrices is a rotation matrix.
- c) The product of a reflection matrix times a rotation matrix is what kind of matrix?

(Hint, use the fact, proved in the next chapter, that  $\det(AB) = (\det A)(\det B)$ .)

**Exercise 2.5.** Find a matrix  $A$  such that  $A^2 = I$ , but which is not a reflection matrix.

**Exercise 2.6.** Find a rotation matrix whose entries are nonzero rational numbers. (A rational number is a quotient of two integers.  $\frac{1}{\sqrt{2}}$  is not a rational number. Hint: Consider the 3-4-5 right triangle. )

**Exercise 2.7.** Suppose  $A$  is a rotation or a reflection. Show that  $A^T = A^{-1}$ .