Linear Algebra Notes

Chapter 3

DETERMINANT AND TRACE

We have seen the determinant of a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\det A = ad - bc.$$

This chapter contains arithmetic and geometric properties of the determinant, and its cousin, the trace (defined below).

Multiplicative Property of Determinant. *If A and B are matrices, then*

$$\det(AB) = \det(A)\det(B).$$

Proof. Say that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix},$$

so

$$\begin{aligned} \det(AB) &= (aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc') \\ &= aa'dd' + bc'cb' - ab'dc' - bd'ca' \\ &= (ad - bc)(a'd' - b'c') \\ &= \det(A)\det(B). \end{aligned}$$

□

For example

$$\det(A^{-1}) = \frac{1}{\det A},$$

because

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

The **trace** of a matrix is the sum of the diagonal entries.

$$\operatorname{tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d.$$

Like the determinant, there are formulas, and non-formulas. A few chapters from now, the trace and determinant will be combined to make the **characteristic polynomial** , which contains all the essential features of a matrix. (The entries in the matrix are not its essential features!)

Exercise 3.1. *Prove or disprove:*

(a) $\operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B)$.
(b) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
(c) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.
(d) $\operatorname{tr}(A^{-1}) = \frac{1}{\operatorname{tr}(A)}$.
(e) $\operatorname{tr}(xA) = x\operatorname{tr}(A)$. *(Here x is a scalar.)*
(f) $\det(xA) = x\det(A)$.
(g) $\det(A + B) = \det(A) + \det(B)$.
(h) $\det(xI - A) = x^2 - \operatorname{tr}(A)x + \det(A)$.