Linear Algebra Notes

Chapter 4

MATRICES AS LINEAR MAPS

A matrix can be used to move points around in the plane, as follows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

Thus, the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ moves the point (x, y) to the point $(ax + by, cx + dy)$.

Different matrices move points around in different ways. For example, a rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotates every point by angle θ in the counterclockwise direction.

The identity matrix I (which is a rotation matrix for $\theta = 0$) sends every point to itself. No point is moved by the identity matrix. The matrix $-I$ is rotation by $\theta = \pi$, so it sends every point to its antipode with respect to the origin.

On the other hand, will see in the next chapter that a reflection matrix moves points by reflecting them about a line through the origin. This is quite different from a rotation, because a reflection fixes (does not move) every point on its reflecting line, while a rotation (except for I) fixes only the origin.

A matrix A is completely determined by what it does to the vectors $(1, 0)$ and $(0, 1)$. These two vectors are used so often that we give them permanent names:

$$\mathbf{e}_1 = (1, 0), \qquad \mathbf{e}_2 = (0, 1).$$

Now, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A\mathbf{e}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}, \qquad \text{and} \qquad A\mathbf{e}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

This simple observation is perhaps the

Most Important Thing. *$A\mathbf{e}_1$ is the first column of A and $A\mathbf{e}_2$ is the second column of A.*

For example, what is the matrix that sends $(1, 0)$ to $(2, 3)$ and $(0, 1)$ to $(4, 1)$? Answer: The matrix is

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}.$$

What is the matrix that reflects about the line $y = x$ and then reflects again about the line $y = -x$? Answer: If you draw a picture, you'll see that $\mathbf{e}_1$ goes to $-\mathbf{e}_1$ and $\mathbf{e}_2$ goes to $-\mathbf{e}_2$. So the matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

Exercise 4.1.

(a) *Find the matrix sends* $(1,0)$ *to* $(1,2)$ *and sends* $(0,1)$ *to* $(2,1)$.
(b) *Find the matrix that sends* $(1,0)$ *to* $(2,1)$ *and sends* $(0,1)$ *to* $(1,1)$.
(c) *Find the matrix that sends* $(2,1)$ *to* $(1,2)$ *and sends* $(1,1)$ *to* $(2,1)$. *Hint for (c): Use (a) and the inverse of the matrix in (b).*

Exercise 4.2. *A octagon has eight vertices, starting at* $(1,0)$ *and rotating by multiples of* $\pi/4$. *Compute the matrix* A *that does this rotation, and then compute* $A,\ A^2,\ldots,A^7$. *Plot the first columns of these matrices. You should get to find the coordinates of the remaining seven vertices of the octagon.*

Exercise 4.3. *Find the matrix that rotates by* $\pi/4$ *and then reflects about the line* $y=x$.

Exercise 4.4. *Find the matrix that rotates by* $3\pi/4$ *and then reflects about the line* $y=x$.

Exercise 4.5. *If you take all the points on a line through the origin, and multiply them by a matrix, you will get another line. Let* $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$. *Describe how* A *moves the lines through the origin. (Hint: Start with the lines through* $\mathbf{e}_1$ *and* $\mathbf{e}_2$, *and then consider a line with slope* $m\neq 0,\infty$.*)*