Linear Algebra Notes

Chapter 5

REFLECTION MATRICES

Let us find the matrix of reflection about a given line ℓ through the origin in the plane. Let θ be the counterclockwise angle between ℓ and the x-axis. Our reflection sends a point in the plane to its mirror image on the opposite side of ℓ, and we want to find the matrix A that does this. Consider the rotation matrix

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The matrix B rotates the x-axis onto ℓ. Let

$$\mathbf{u} = B\mathbf{e}_1, \qquad \mathbf{v} = B\mathbf{e}_2.$$

Thus, $\mathbf{u}$ is the first column of B; it lies on ℓ. And $\mathbf{v}$ is the second column of B; it is perpendicular to ℓ, and points 90 degrees counterclockwise from $\mathbf{u}$.

Since $\mathbf{u}$ lies on ℓ, we have $A\mathbf{u} = \mathbf{u}$. Since $\mathbf{v}$ is perpendicular to ℓ, we have $A\mathbf{v} = -\mathbf{v}$. Since $B\mathbf{e}_1 = \mathbf{u}$ and $B\mathbf{e}_2 = \mathbf{v}$, we get

$$AB\mathbf{e}_1 = B\mathbf{e}_1 \quad \text{and} \quad AB\mathbf{e}_2 = -B\mathbf{e}_2.$$

Applying B^{-1}, we have

$$B^{-1}AB\mathbf{e}_1 = \mathbf{e}_1 \quad \text{and} \quad B^{-1}AB\mathbf{e}_2 = -\mathbf{e}_2.$$

This means that

$$B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4b}$$

Now multiply on the left by B, and on the right by B^{-1}, and get

$$\begin{aligned} A &= B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} B^{-1} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}. \end{aligned}$$

In summary, we have shown that the reflection about the line ℓ having angle θ with respect to the x-axis has matrix

$$A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Note that you could write A as a product of a rotation times a very simple reflection:

$$A = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This says that reflection about the line with angle θ is the same as reflection about the x-axis followed by rotation by 2θ.

The key step in the computation of A was equation (4b). The idea in this computation is that A behaves simply with respect to $\mathbf{u}, \mathbf{v}$, but not simply with respect to $\mathbf{e}_1, \mathbf{e}_2$. So we take the matrix B sending $\mathbf{e}_1, \mathbf{e}_2$ to $\mathbf{u}, \mathbf{v}$, then A acts simply, and then we go back to $\mathbf{e}_1, \mathbf{e}_2$ by means of B^{-1}. This means the matrix $B^{-1}AB$ will be simple (that is equation (4b)), and then we multiply by B on the left, B^{-1} on the right, to extract A.

The remarkable thing is that the same idea works for almost any matrix, even if it is not a reflection. That is, almost any matrix has a favorite pair of vectors $\mathbf{u}, \mathbf{v}$, called "eigenvectors", on which the matrix behaves simply. These vectors are usually hidden from view, but there is a way to find them, and then use them to compute and analyze the matrix. This is the subject of the next few chapters.

Exercise5.1. *A hexagon has six vertices, starting at* $(1, 0)$ *and rotating by multiples of* $\pi/3$.

(a) *Find the coordinates of the remaining five vertices.*
(b) *There are six reflections that map the hexagon to itself. Draw the reflecting lines of these reflections and find their matrices.*

Exercise 5.2. *Suppose* A *is reflection matrix about a line with angle* θ*, as above, and* A' *is a reflection about a line with angle* ϕ*. Then* $A'A$ *is a rotation matrix. What is the angle of rotation of* $A'A$*? Check your answer by taking* A, A' *to be two reflections of the hexagon, as in exercise 5.1b.*

Exercise 5.3. *Find two reflection matrices that do not commute with each other.*

Exercise 5.4. *Suppose* A *and* A' *are two distinct reflections that commute with each other. What is the relation between their reflecting lines? (Hint: Compare* AA' *and* $A'A$*, which were computed in exercise 5.2.)*