Linear Algebra Notes

Chapter 6

FIBONACCI NUMBERS

The Fibonacci numbers are

$$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8, \cdots.$$

To get the next Fibonacci number, you add the previous two. Thus, Fibonacci numbers are defined by the recursive formula

$$F_0 = 0,\ F_1 = 1, \quad F_{n+1} = F_{n-1} + F_n.$$

To compute, say, F_{100} with this formula, you need to first compute F_{99} and F_{98}, and for these you need to compute F_{97} and F_{96}, and so on. Is there a formula that will tell us F_{100} without having to compute all the previous Fibonacci numbers? Yes. We'll find it with matrices, using the same principle that we used to find the matrix of a reflection.

Let

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(The Phibonacci matrix!) Then

$$\Phi \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} + F_n \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

Thus, applying Φ to the vector whose components are a pair of successive Fibonnaci numbers, we get the vector whose components are the next pair of successive Fibonnaci numbers. For example,

$$\begin{aligned} \Phi \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \\ \Phi^2 \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} &= \Phi \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}, \\ \Phi^3 \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} &= \Phi \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_4 \end{bmatrix}, \quad \text{etc.} \end{aligned}$$

In general,

$$\Phi^n \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

But $F_0 = 0$ and $F_1 = 1$, so this just says that

$$\Phi^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

This vector is the second column of Φ^n. In fact,

$$\Phi^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

Therefore, if we want to compute F_{100}, "all" we have to do is compute Φ^{100}.

As it stands, this is just a rephrasing of the problem, and we have not achieved much. If Φ were a diagonal matrix, then it would be easy to compute Φ^{100}, but Φ is not a diagonal matrix. However, recall what happened with the reflection matrix A, in chapter 5. There was another matrix B such that $B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is a diagonal matrix. So maybe there is a matrix B such that $B^{-1}\Phi B$ is a diagonal matrix. Would that help us? Say that we find such a B, and that

$$B^{-1}\Phi B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

Then

$$(B^{-1}\Phi B)^n = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}^n = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix}. \tag{6a}$$

But notice also that

$$(B^{-1}\Phi B)^n = (B^{-1}\Phi B)(B^{-1}\Phi B)(B^{-1}\Phi B)\cdots(B^{-1}\Phi B) = B^{-1}\Phi^n B. \tag{6b}$$

Combining equations (6a) and (6b), we get

$$B^{-1}\Phi^n B = \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix}.$$

As we did with the reflection matrix, we extract Φ by multiplying on the left by B and on the right by B^{-1}. This gives

$$\Phi^n = B \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} B^{-1}. \tag{6c}$$

And this is a formula for Φ^n, which we have seen gives a formula or F_n. So we can solve the problem if we can find a matrix B such that $B^{-1}\Phi B$ is diagonal.

We shall find B using a little magic, which will be explained later (the explanation is rather long).

For any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define the **characteristic polynomial of** A to be

$$P_A(x) = x^2 - (a+d)x + (ad - bc).$$

For the matrix Φ, we get

$$P_\Phi(x) = x^2 - x - 1.$$

The roots of $P_\Phi(x)$ are

$$\frac{1}{2}(1 \pm \sqrt{5}).$$

We take one root to be λ, and the other to be μ, say

$$\lambda = \frac{1}{2}(1+\sqrt{5}) \qquad \mu = \frac{1}{2}(1-\sqrt{5}).$$

These numbers are called the **Eigenvalues** of Φ. They will turn out to be the diagonal entries of $B^{-1}\Phi B$ (once we find B). Because λ and μ are the roots of $x^2 - x - 1$, we have

$$\lambda^2 = \lambda + 1, \qquad \mu^2 = \mu + 1.$$

Now define vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ \mu \end{bmatrix}.$$

Note that

$$\Phi\mathbf{u} = \begin{bmatrix} \lambda \\ 1+\lambda \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \lambda\mathbf{u},$$

so

$$\Phi\mathbf{u} = \lambda\mathbf{u}, \tag{6d}$$

and likewise

$$\Phi\mathbf{v} = \mu\mathbf{v}. \tag{6e}$$

The vectors $\mathbf{u}$ and $\mathbf{v}$ are called **Eigenvectors** of Φ.

Let B be the matrix whose first column is $\mathbf{u}$ and whose second column is $\mathbf{v}$. In other words,

$$B = \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}.$$

As always, this means that $B\mathbf{e}_1 = \mathbf{u}$ and $B\mathbf{e}_2 = \mathbf{v}$.

We must check that $B^{-1}\Phi B$ is diagonal. We could compute B^{-1} and just multiply all three matrices, but it is easier to proceed as we did with the reflection matrix, and calculate the effect of $B^{-1}\Phi B$ on the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$. Using equation (6d), we get

$$B^{-1}\Phi B\mathbf{e}_1 = B^{-1}\Phi\mathbf{u} = B^{-1}(\lambda\mathbf{u}) = \lambda B^{-1}\mathbf{u} = \lambda\mathbf{e}_1,$$

and likewise, using (6e), we get

$$B^{-1}\Phi B\mathbf{e}_1 = \mu\mathbf{e}_2.$$

Thus,

$$B^{-1}\Phi B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

Rejoice, we conquer! Now, remember, we wanted to compute Φ^n. Equation (6c) says that

$$\Phi^n = B \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} B^{-1} = \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}^{-1}.$$

Now $\det \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix} = -1/\sqrt{5}$, so

$$\begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} -\mu & 1 \\ \lambda & -1 \end{bmatrix},$$

so we get, finally,

$$\begin{aligned} \Phi^n &= \begin{bmatrix} 1 & 1 \\ \lambda & \mu \end{bmatrix} \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -\mu & 1 \\ \lambda & -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \mu^n\lambda - \lambda^n\mu & \lambda^n - \mu^n \\ \mu^{n+1}\lambda - \lambda^{n+1}\mu & \lambda^{n+1} - \mu^{n+1} \end{bmatrix}. \end{aligned}$$

Since $\lambda\mu = -1$, we can write this more simply as

$$\Phi^n = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda^{n-1} - \mu^{n-1} & \lambda^n - \mu^n \\ \lambda^n - \mu^n & \lambda^{n+1} - \mu^{n+1} \end{bmatrix}.$$

Recall that the upper right entry of Φ^n is F_n, so we get

$$F_n = \frac{1}{\sqrt{5}}(\lambda^n - \mu^n). \qquad \text{(Fibonacci Formula)}$$

Recall that

$$\lambda = \frac{1}{2}(1+\sqrt{5}) = 1.618\ldots.$$

and

$$\mu = \frac{1}{2}(1-\sqrt{5}) = -0.618\ldots.$$

If n is large in the Fibonacci Formula (which is the case we are interested in), then μ^n will be small, so F_n will be close to $\frac{1}{\sqrt{5}}(\lambda^n)$.

This gives a practical method for computing F_n for large n: First compute $\frac{1}{\sqrt{5}}(\lambda^n)$, then take the nearest integer.

For example, to compute F_{100} without computing any other Fibonacci numbers: We first compute

$$\frac{\lambda^{100}}{\sqrt{5}} = 3.54224848179261915075000000 \times 10^{20}.$$

We need enough decimal places in order to tell what the nearest integer is; in this case we need at least 20 decimal places (to be safe, I took 25). So

$$F_{100} = 3.54224848179261915075 \times 10^{20} = 354224848179261915075.$$

Exercise 6.1.

(a) *Use this method to compute F_{20}, F_{21} and F_{30} without computing any other Fibonnaci numbers.*

(b) *Compute F_{30} again by starting with your values for F_{20} and F_{21} from part (a) and then computing $F_{22},\ F_{23}, \ldots, F_{30}$ using the recursive formula $F_{n+1} = F_{n-1} + F_n$.*

Exercise 6.2. *Show that*

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lambda.$$

Exercise 6.3. *The eigenvectors* $\mathbf{u}, \mathbf{v}$ *of* Φ *live on the lines with equations* $y = \lambda x$, $y = \mu x$, *respectively. Consider the function*

$$f(x, y) = (y - \lambda x)(y - \mu x)$$

which is zero on these lines.

(a) *Show that* $f(x, y) = y^2 - xy - x^2$.

(b) *Given a point* (x, y), *define* (x', y') *by the equation*

$$\Phi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Show that

$$f(x', y') = -f(x, y).$$

(c) *Let* C_+ *be the graph of the equation* $y^2 - xy - x^2 = 1$, *and let* C_- *be the graph of the equation* $y^2 - xy - x^2 = -1$. *These graphs are hyperbolas, having the lines* $y = \lambda x$, $y = \mu x$ *as asymptotes. Use part (b) to show that the point* (F_n, F_{n+1}) *lives on* C_+ *if* n *is even and on* C_- *if* n *is odd.*

A picture to accompany this exercise will be drawn in class.

Exercise 6.4. *The "Lucas numbers" are defined like the Fibonaccis, except the two starting values are* $L_0 = 2$, $L_1 = 1$. *Thus,* $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, *etc. Using the method of this chapter, find a formula for* L_n *that does not require computing any other Lucas numbers. (Note the same matrix* Φ *is used, but the initial vector is different).*