Chapter 7

MIGRATION

This is another example illustrating the use of eigenvalues for finding powers of a matrix.

Suppose we have a population of P students, who live either in dorms or apartments. Many students are not happy with their accomodations, so they migrate to the other kind of living space. Each month, certain fixed proportions of the student population will migrate. Let

$$\begin{aligned}
a &= \text{proportion of students in apts who stay in apts}\\
b &= \text{proportion of students in dorms who move to apts}\\
c &= \text{proportion of students in apts who move to dorms}\\
d &= \text{proportion of students in dorms who stay in dorms.}
\end{aligned}$$

Being proportions, the numbers a, b, c, d are between 0 and 1. Moreover, since all students in apts must either stay or move, and likewise for all students in dorms, we have

$$a + c = b + d = 1.$$

If $b = c = 0$ then no one moves. If $b = c = 1$ then everyone in the apts moves to the dorms and vice-versa, and they just keep switching back and forth every month. These situations are easy to understand, so from now on we assume that b, c are not both 0, and not both 1. (One of them could be 0, but not the other, and one of them could be 1, but not the other.)

Our goal, as Directors of University Housing, is to predict the long term distribution of students into apts and dorms. Will the populations keep fluctuating? Will all of the students end up in one place or the other? Will we reach some other stable equilibrium? Maybe we Directors have control over the initial distribution of students. Can we choose this initial distribution to minimize the fluctuations caused by student migration?

Let

$$\begin{aligned}
x_0 &= \text{initial number of students in apartments,}\\
y_0 &= \text{initial number of students in dorms.}
\end{aligned}$$

Thus we have the initial distribution vector

$$\mathbf{w}_0 = (x_0, y_0).$$

Note that

$$x_0 + y_0 = P,$$

since P is the total number of students. At the end of the first month, the number of students in apartments is

$$x_1 = ax_0 + by_0$$

and the number of students in dorms is

$$y_1 = cx_0 + dy_0.$$

In other words,

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

The following month, the distribution $\mathbf{w}_2 = (x_2, y_2)$ will be given by

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

So we have a Migration Matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and after n months, the distribution vector $\mathbf{w}_n = (x_n, y_n)$ is given by

$$\mathbf{w}_n = M^n \mathbf{w}_0. \tag{7a}$$

Equation (7a) is called a **discrete dynamical system** . Any matrix, like M, with non-negative entries, and whose columns add up to one, is called a **stochastic matrix** . The forthcoming analysis will apply to any discrete dynamical system involving a stochastic matrix.

To calculate M^n, we'll find the eigenvalues, as before. But first, since $a = 1 - c$ and $d = 1 - b$, we can rewrite M as

$$M = \begin{bmatrix} 1-c & b \\ c & 1-b \end{bmatrix}.$$

The characteristic polynomial of M is

$$P_M(t) = t^2 + (b + c - 2)t + 1 - b - c = (t - [1 - b - c])(t - 1),$$

so the eigenvalues of M are

$$\lambda = 1 - b - c, \qquad \text{and} \qquad \mu = 1.$$

The eigenvectors are

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} b \\ c \end{bmatrix}.$$

I haven't yet told you how to find these vectors (next chapter), but we can check that they are indeed eigenvectors, because

$$M\mathbf{u} = \begin{bmatrix} 1-b-c \\ c-1+d \end{bmatrix} = (1-b-c) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda \mathbf{u},$$

and

$$M\mathbf{v} = \begin{bmatrix} b - bc + bc \\ bc + c - bc \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix} = \mu \mathbf{v}.$$

The change of basis matrix is

$$B = \begin{bmatrix} 1 & b \\ -1 & c \end{bmatrix},$$

and its determinant is $\det B = b + c$, which is nonzero since we assumed b, c are nonnegative and not both zero. Therefore B is invertible, and

$$B^{-1} = \frac{1}{b+c} \begin{bmatrix} c & -b \\ 1 & 1 \end{bmatrix}.$$

Recall, our initial distribution vector is

$$\mathbf{w}_0 = (x_0, y_0).$$

We write $\mathbf{w}_0$ in terms of the eigenvectors $\mathbf{u}, \mathbf{v}$, by applying B^{-1} to $\mathbf{w}_0$, and get

$$B^{-1}\mathbf{w}_0 = \frac{1}{b+c} \begin{bmatrix} c & -b \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{b+c} \begin{bmatrix} cx_0 - by_0 \\ x_0 + y_0 \end{bmatrix} = \frac{1}{b+c} \begin{bmatrix} cx_0 - by_0 \\ P \end{bmatrix}.$$

Therefore

$$\mathbf{w}_0 = \left(\frac{cx_0 - by_0}{b+c} \right) \mathbf{u} + \left(\frac{P}{b+c} \right) \mathbf{v}.$$

After n months, the distribution vector will be

$$M^n \mathbf{w}_0 = \left(\frac{cx_0 - by_0}{b+c} \right) M^n \mathbf{u} + \left(\frac{P}{b+c} \right) M^n \mathbf{v}.$$

But recall that $M\mathbf{u} = \lambda \mathbf{u}$. That means

$$M^2 \mathbf{u} = M(\lambda \mathbf{u}) = \lambda M \mathbf{u} = \lambda^2 \mathbf{u},$$

and in general,

$$M^n \mathbf{u} = \lambda^n \mathbf{u}, \qquad \text{and} \qquad M^n \mathbf{v} = \mu^n \mathbf{v}.$$

So we get

$$M^n \mathbf{w}_0 = \left(\frac{cx_0 - by_0}{b+c} \right) \lambda^n \mathbf{u} + \left(\frac{P}{b+c} \right) \mu^n \mathbf{v}. \tag{7b}$$

Forget about the expressions inside the $\Big(\ \Big)$'s for the moment, since they don't involve n. Concentrate on $\lambda^n \mathbf{u}$ and $\mu^n \mathbf{v}$. First of all, $\mu = 1$, so

$$\mu^n \mathbf{v} = \mathbf{v}.$$

Next, recall that $\lambda = 1 - b - c$, which is less than 1, since b, c are not both zero, and bigger than -1, since b, c are not both 1. That is,

$$-1 < \lambda < 1.$$

This means that $\lambda^n \to 0$ as n gets large.

This is the crucial deduction. It means that

$$M^n \mathbf{w}_0 \sim \left(\frac{P}{b+c}\right)\mathbf{v} \qquad \text{(approximately, for large } n\text{)}$$

Thus the population distribution reaches a **stable equilibrium point**

$$\mathbf{w}_\infty = \left(\frac{P}{b+c}\right)\mathbf{v} = (\frac{Pb}{b+c}, \frac{Pc}{b+c}).$$

As Directors of Housing, we could design dorms to handle this population distribution (for a given P), and after some months of under and over-crowding, the size of our dorms would be just about right.

Notice also that $\mathbf{w}_\infty$ is a scalar multiple of $\mathbf{v}$, so

$$M\mathbf{w}_\infty = \mathbf{w}_\infty.$$

Hence, if we could choose the initial distribution vector $\mathbf{w}$ to be $\mathbf{w}_\infty$, then the distribution would not fluctuate at all, even as students move back and forth. If $\mathbf{w}_\infty$ involves fractions, then we would choose $\mathbf{w}$ with integer coordinates as close as possible to $\mathbf{w}_\infty$, and then the distribution would only fluctuate a little bit.

A picture to go along with this analysis will be drawn in class.

Exercise 7.1. *You may have noticed that equation (7b) gives a formula for M^n, but it is not in the form of a matrix. Calculate the matrix M^n.*

Exercise 7.2. *Suppose $P = 1000$ students, and that each month, 80% of students in dorms switch to apts, and 60% of students in apts switch to dorms.*

(a) *Find the migration matrix M and its eigenvalues.*
(b) *Assume that, initially, 500 students are in dorms and the other half are in apts. Find the population distribution at months 1,2,3,4 (round to the nearest integer).*
(c) *Find the stable equilibrium point $\mathbf{w}_\infty$.*
(d) *Suppose that initially all 1000 students are in the dorms. How many months will it take to reach the stable equilibrium point? (In other words, find the smallest n such that the entries of $M^n\mathbf{w}_0$ differ from those of $\mathbf{w}_\infty$ by less than 1.)*
(e) *As Housing Director, you crave equilibrium (otherwise, either your facilities are strained, or your voicemail is clogged with complaints about students from residents of Allston-Brighton). From this point of view, is the initial distribution in (d) the worst-case scenario, or are there other initial distributions that could take even more months to reach equilibrium?*