Linear Algebra Notes

Chapter 8

EIGENVALUES AND EIGENVECTORS

In previous chapters, we have seen a few uses for the eigenvalues and eigenvectors of a matrix, but I have not explained how to find the eigenvectors. In this chapter, we will learn a general method for finding eigenvectors.

Take a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

An **eigenvector** of A is a *nonzero* vector $\mathbf{u}$ such that $A\mathbf{u}$ is a scalar multiple of $\mathbf{u}$. The scalar is called the **eigenvalue** of $\mathbf{u}$. If λ denotes this scalar, then we have

$$A\mathbf{u} = \lambda\mathbf{u}. \tag{8a}$$

Any vector $\mathbf{u}$ that satisfies equation (8a) is an eigenvector, with eigenvalue λ.

Often it is helpful to scale the eigenvectors. Note that if $A\mathbf{u} = \lambda\mathbf{u}$, and t is any number, then

$$A(t\mathbf{u}) = tA\mathbf{u} = t\lambda\mathbf{u} = \lambda(t\mathbf{u}),$$

so $t\mathbf{u}$ also satisfies equation (8a). Thus, we can multiply any eigenvector by a nonzero number and get another eigenvector with the same eigenvalue.

The simplest case is when A is a scalar matrix, like

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

Then $A\mathbf{u} = 5\mathbf{u}$ for every vector $\mathbf{u}$, so every nonzero vector in $\mathbb{R}^2$ is an eigenvector of A, with eigenvalue $\lambda = 5$.

The next simplest case is when A is a non-scalar diagonal matrix, like

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}.$$

Then $\mathbf{e}_1$ is an eigenvector with eigenvalue 5, and $\mathbf{e}_2$ is an eigenvector with eigenvalue 7.

The next simplest case is when A is an upper-triangular matrix, like

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 7 \end{bmatrix}.$$

Then $\mathbf{e}_1$ is an eigenvector with eigenvalue 5, but $\mathbf{e}_2$ is *not* an eigenvector with eigenvalue 7. It is still true that 7 is an eigenvalue of A, but the corresponding eigenvector is not obvious (see first example below).

Here is a method for computing eigenvalues and eigenvectors. It only applies to matrices with two distinct eigenvalues (which is what we usually find). Later, I will tell you about matrices with the same eigenvalue repeated twice.

The eigenvalues are computed first: They are the roots of the characteristic polynomial

$$P_A(x) = x^2 - (a+d)x + (ad - bc).$$

Let λ and μ be the roots of $P_A(x)$. We assume $\lambda \neq \mu$. For $\mathbf{u}$, we can take

$$\mathbf{u} = \begin{bmatrix} b \\ \lambda - a \end{bmatrix},$$

if this is not the zero vector. Occasionally this formula would give $\mathbf{u} = (0,0)$, which is unacceptable, but that could only happen if if $b = 0 = \lambda - a$. In this case, take

$$\mathbf{u} = \begin{bmatrix} \lambda - d \\ c \end{bmatrix}.$$

To see that this works, recall that

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0, \quad \text{so} \quad \lambda^2 - a\lambda = d\lambda - (ad - bc).$$

Therefore

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b \\ \lambda - a \end{bmatrix} &= \begin{bmatrix} ab + b(\lambda - a) \\ cb + d(\lambda - a) \end{bmatrix} \\ &= \begin{bmatrix} b\lambda \\ \lambda^2 - a\lambda \end{bmatrix} \\ &= \lambda \begin{bmatrix} b \\ \lambda - a \end{bmatrix}. \end{aligned}$$

So the vector $\lambda \begin{bmatrix} b \\ \lambda - a \end{bmatrix}$ satisfies equation (8a). Likewise,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda - d \\ c \end{bmatrix} = \lambda \begin{bmatrix} \lambda - d \\ c \end{bmatrix},$$

so the vector $\begin{bmatrix} \lambda - d \\ c \end{bmatrix}$ also satisfies equation (8a). To find the μ-eigenvector, do the same thing with λ replaced by μ. That is, take

$$\mathbf{v} = \begin{bmatrix} b \\ \mu - a \end{bmatrix},$$

if this is not the zero vector, otherwise (i.e. if $b = 0 = \mu - a$), take

$$\mathbf{v} = \begin{bmatrix} \mu - d \\ c \end{bmatrix}.$$

First example: Suppose

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 7 \end{bmatrix}.$$

The characteristic polynomial is

$$P_A(x) = x^2 - 12x + 35 = (x-5)(x-7).$$

The eigenvalues are

$$\lambda = 5, \quad \mu = 7.$$

The eigenvectors are

$$\mathbf{u} = \begin{bmatrix} 3 \\ 5-5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$
$$\mathbf{v} = \begin{bmatrix} 3 \\ 7-5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

If we want, we can scale the first eigenvector $\mathbf{u}$ and replace it by $\mathbf{u} = \mathbf{e}_1$. The eigenvector $\mathbf{v}$ is the non-obvious eigenvector with eigenvalue 7.

Second example: Let

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix}.$$

The characteristic polynomial is

$$P_A(x) = x^2 - 8x + 15 = (x-3)(x-5).$$

The eigenvalues are

$$\lambda = 3, \quad \mu = 5.$$

The eigenvectors are

$$\mathbf{u} = \begin{bmatrix} 4 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$
$$\mathbf{v} = \begin{bmatrix} 4 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

We can scale our eigenvectors, and take instead

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Just to make sure, we could check them:

$$A\mathbf{u} = \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\mathbf{u},$$

and similarly, $A\mathbf{v} = 5\mathbf{v}$.

Third example: Take the migration matrix

$$M = \begin{bmatrix} 1-c & b \\ c & 1-b \end{bmatrix}.$$

In the previous chapter, we computed

$$\lambda = 1 - b - c, \qquad \mu = 1.$$

The recipe above gives, for the λ-eigenvector,

$$\begin{bmatrix} b \\ 1-b-c-(1-c) \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix},$$

which I scaled to $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For the other eigenvector, the recipe gives

$$\mathbf{v} = \begin{bmatrix} b \\ 1-(1-c) \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix},$$

and there was no point in scaling it.

Now, once you have the eigenvalues λ, μ with corresponding eigenvectors $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, you can make the **change of basis matrix**

$$B = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

This matrix B will be invertible, because $\mathbf{u}$ and $\mathbf{v}$ are non-proportional. (If they were proportional, they would have the same eigenvalue but we are assuming $\lambda \neq \mu$!) Just as in the previous chapters, we then have

$$B^{-1}AB = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

so

$$A = B \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} B^{-1},$$

and

$$A^n = B \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix} B^{-1}.$$

In the second example above with

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix},$$

we have

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

and sure enough,

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The same thing would happen if you didn't scale the eigenvectors. The scalars would cancel out.

Exercise 8.1. *Find the eigenvalues and eigenvectors of the following matrices. (You can and should check your answers.)*

a)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 3 & 4 \\ 2 & -3 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 3 & 0 \\ 2 & -3 \end{bmatrix}$$

d)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

e)

$$A = \begin{bmatrix} 15 & -10 \\ 21 & -14 \end{bmatrix}$$

Exercise 8.2. *Suppose A is a reflection matrix*

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad a^2 + b^2 = 1.$$

a) *Find the eigenvalues and eigenvectors of A.*
b) *Find a vector on the reflecting line of A.*
c) *Find a vector perpendicular to the reflecting line of A.*
All of your answers will involve a and b.

Exercise 8.3. *Suppose A is a matrix whose entries you do not know, but you do know that its eigenvalues are $\lambda = 2$ and $\mu = 3$, with corresponding eigenvectors $\mathbf{u} = (1, 3)$, $\mathbf{v} = (6, -1)$.*

a) Find A.

b) Find the eigenvalues and eigenvectors of A^{100}.

Exercise 8.4. *Let*

$$A = \begin{bmatrix} 16 & -10 \\ 21 & -13 \end{bmatrix}.$$

Find an explicit formula for A^n.