

Note 2

Line integrals in the plane, vector fields, work integrals, conservative vector fields and independence of path

1. Line Integrals in the Plane

Given a path in the plane $\mathbf{c}(t) = (x(t), y(t))$, for $a \leq t \leq b$, and a pair of functions $P(x, y), Q(x, y)$, the line integral

$$\int_{\mathbf{c}} P dx + Q dy$$

is defined to be

$$\int_{\mathbf{c}} P dx + Q dy = \int_a^b [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)] dt.$$

In other words,

$$dx = x'(t) dt, \quad dy = y'(t) dt,$$

and you put $x = x(t), y = y(t)$ into $P(x, y), Q(x, y)$, then integrate the resulting function of t as usual.

Example 1: Let $\mathbf{c}(t) = (a + r \cos t, b + r \sin t)$, $0 \leq t \leq 2\pi$. Thus, \mathbf{c} is a counterclockwise circle of radius r , centered at (a, b) . Take $P = 0, Q = x$. Then the line integral is

$$\int_{\mathbf{c}} x dy = \int_0^{2\pi} (a + r \cos t)(r \cos t) dt = \pi a^2.$$

In the next example, the path \mathbf{c} is given in segments; the line integral over \mathbf{c} is the sum of the integrals over the segments.

Example 2: Let \mathbf{c} be the counterclockwise boundary of the rectangle with vertices $(0, 0), (l, 0), (l, h), (0, h)$. Since $y' = 0$ on the horizontal sides, and $x = 0$ on the left vertical side, and the right vertical side is parametrized by (l, t) , $0 \leq t \leq h$, we have

$$\int_{\mathbf{c}} x dy = \int_0^h l dt = hl.$$

Note that in Examples 1,2, the integral of $x dy$ over a closed counterclockwise curve equals the area inside the curve.

Exercise 1.1 Compute $\int_{\mathbf{c}} x dy$ over the following counterclockwise curves, and then compute the area inside of each curve.

a) \mathbf{c} is the triangle from $(0, 0)$ to $(a, 0)$ to $(0, b)$ back to $(0, 0)$.

b) \mathbf{c} follows the graph of $y = x^2$ from $(0, 0)$ to $(1, 1)$, then follows the graph of $x = y^2$ from $(1, 1)$ back to $(0, 0)$.

Exercise 1.2 Compute $\int_{\mathbf{c}} x dx + y dy$ along the curves

$$\mathbf{c}_1(t) = (t, t^2), \quad \mathbf{c}_2(t) = (t, t), \quad \mathbf{c}_3(t) = (t^2, t),$$

all with $0 \leq t \leq 1$. You should get the same answer each time.

Exercise 1.3 Compute $\int_c -y \, dx + x \, dy$ along the three curves in the previous problem. You will get different answers this time.

2. Vector Fields

The physical interpretation of line integrals is in terms of vector fields. The scalar-valued functions $P(x, y), Q(x, y)$ together determine a vector field

$$\mathbf{F}(x, y) = (P(x, y), Q(x, y)).$$

Geometrically, \mathbf{F} attaches a vector to every point in the xy -plane. For example, at the point $(1, 2)$, \mathbf{F} attaches the vector

$$\mathbf{F}(1, 2) = (P(1, 2), Q(1, 2)).$$

Physically, \mathbf{F} represents a force at each point (eg gravity or electric) or velocity of a fluid (eg wind), or temperature gradient...

Example 1:

$$(2.1) \quad \mathbf{F}(x, y) = (x, y) \quad \text{Explosion}$$

This vector field attaches the *vector* (x, y) to the *point* (x, y) . So it is exploding. Note that $\mathbf{F} = \nabla f$, where

$$f(x, y) = \frac{1}{2}(x^2 + y^2).$$

The arrows of \mathbf{F} are therefore normal to the level curves of f , which are circles centered at $(0, 0)$. This vector field occurred in exercise 1.2 above.

Vector fields which are gradients of functions are called **conservative vector fields**.

Example 2:

$$(2.2) \quad \mathbf{F}(x, y) = (-y, x) \quad \text{Counterclockwise spin}$$

This vector field, which occurred in exercise 1.3 above, is Perpendicular to the field in (2.1). The vector field $\mathbf{F}(x, y) = (-y, x)$ is not the gradient of any function. For if it were, then there would be a function $g(x, y)$ so that $(-y, x) = \nabla g = (g_x, g_y)$. This would mean that

$$g_{xy} = \frac{\partial}{\partial y}(g_x) = \frac{\partial}{\partial y}(-y) = -1,$$

and

$$g_{yx} = \frac{\partial}{\partial x}(g_y) = \frac{\partial}{\partial x}(x) = +1.$$

But $g_{xy} = g_{yx}$, so this cannot happen. Therefore \mathbf{F} is not a conservative vector field.

Example 3: Take a fixed vector \mathbf{v} , and let

$$(1.3) \quad \mathbf{F}(x, y) = \mathbf{v} \quad \text{Constant vector field}$$

This vector field attaches the same vector \mathbf{v} at every point. This represents, for example, a steady wind velocity, with the same direction and magnitude at every point.

3. Work Integrals

Given a vector field $\mathbf{F} = (P, Q)$, and a path $\mathbf{c}(t) = (x(t), y(t))$, $a \leq t \leq b$, the integral of \mathbf{F} along C is defined to be

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

That is, you plug $\mathbf{c}(t)$ into \mathbf{F} , take dot product with the velocity $\mathbf{c}'(t)$, and integrate the resulting function of t as usual. This integral measures how much work is done by F on a particle moving with position $\mathbf{c}(t)$. You can also think of the integral as measuring how much $\mathbf{c}(t)$ is flowing in the direction of F . If we write it out, we get

$$\begin{aligned} \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\ &= \int_a^b (P(x(t), y(t)), Q(x(t), y(t))) \cdot (x'(t), y'(t)) dt \\ &= \int_a^b [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)] dt \\ &= \int_{\mathbf{c}} P dx + Q dy. \end{aligned}$$

So $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c}$ is the same thing as $\int_{\mathbf{c}} P dx + Q dy$. This means you can visualize $\int_{\mathbf{c}} P dx + Q dy$ as the work done by the vector field $F = (P, Q)$ on a particle moving along \mathbf{c} .

Example 1:

$$\mathbf{F}(x, y) = (-y, x), \quad \mathbf{c}(t) = (1 + t, 1 - t^2), \quad 0 \leq t \leq 1.$$

We have

$$\mathbf{F}(\mathbf{c}(t)) = (t^2 - 1, 1 + t) \quad \mathbf{c}'(t) = (1, -2t),$$

so

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_0^1 (t^2 - 1, 1 + t) \cdot (1, -2t) dt = \int_0^1 t^2 - 1 + (1 + t)(-2t) dt = -\frac{7}{3}.$$

The negative answer means that the path \mathbf{c} is generally going against the vector field \mathbf{F} .

Example 2:

$$\mathbf{F}(x, y) = (xy, x^2), \quad \mathbf{c}(t) = \text{line segment from } (1, 2) \text{ to } (2, 1).$$

First,

$$\mathbf{c}(t) = (1, 2) + t(2 - 1, 1 - 2) = (1 + t, 2 - t), \quad \mathbf{c}'(t) = (1, -1), \quad 0 \leq t \leq 1.$$

Now

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} &= \int_0^1 ((1+t)(2-t), (1+t)^2) \cdot (1, -1) dt \\ &= \int_0^1 (1+t)(2-t) - (1+t)^2 dt = -\frac{1}{6}.\end{aligned}$$

Draw a picture and you'll see that \mathbf{c} is going against \mathbf{F} .

Exercise 3.1 Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c}$ for

- a) $\mathbf{F} = (x, y)$, \mathbf{c} is the line segment from $(1, 0)$ to $(0, 2)$
- b) $\mathbf{F} = (x, y)$, $\mathbf{c}(t) = (\cos t, 2 \sin t)$, $0 \leq t \leq \frac{\pi}{2}$.
- c) $\mathbf{F} = (-y, x)$, \mathbf{c} as in a).
- d) $\mathbf{F} = (-y, x)$, \mathbf{c} as in b).

(You should get the same answers for a) and b), but different answers for c) and d). This is connected to the fact that $\mathbf{F} = (x, y)$ is conservative, while $\mathbf{F} = (-y, x)$ is not conservative. See also exercises 1.2,3 above.)

Exercise 3.2 Compute $\int_{\mathbf{c}} F \cdot d\mathbf{c}$ for $F = (3x - 4y, 4x + 2y)$, where \mathbf{c} is the counterclockwise ellipse with equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Exercise 3.3 The World's Most Interesting Vector Field is

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

Compute $\int_{\mathbf{c}} F \cdot d\mathbf{c}$ over the following paths \mathbf{c} :

- a) $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq \frac{\pi}{2}$
- b) The line segment from $(1, 0)$ to $(1, 1)$ followed by the line segment from $(1, 1)$ to $(0, 1)$
- c) $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$ (upper semicircle)
- d) $\mathbf{c}(t) = (\cos t, -\sin t)$, $0 \leq t \leq \pi$ (lower semicircle).

You should get the same answers for a) and b), but different answers for c) and d). What do you guess is the cause of this different behavior?

4. Conservative Vector Fields and Independence of Path

Suppose your vector field $\mathbf{F} = (P, Q)$ is the gradient of some function $f(x, y)$. This means that

$$P = f_x, \quad Q = f_y.$$

Take a path $\mathbf{c}(t)$, $a \leq t \leq b$, and consider the values of f on this path.

Recall the Chain Rule:

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t).$$

Plug this into the line integral:

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} &= \int_{\mathbf{c}} \nabla f \cdot d\mathbf{c} \\ &= \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{c}(t)) dt \quad \text{by the chain rule} \\ &= f(\mathbf{c}(b)) - f(\mathbf{c}(a)) \quad \text{by the fundamental theorem of calculus.}\end{aligned}$$

This means the line integral of ∇f is the difference of the values of f at the endpoints of \mathbf{c} . In other words,

Corollary 1. *The line integral of ∇f over any curve from the point A to the point B is equal to $f(B) - f(A)$.*

So the line integral doesn't care how you get from A to B . In that sense it is "Independent of path". This explains the coincidence of the answers in the earlier exercises. In particular, if \mathbf{c} is a *closed* path, meaning $A = B$, then we have

Corollary 2. *The line integral of a gradient over any closed path is zero.*

Exercise 4.1 Compute

$$\int_{\mathbf{c}} \cos x \cos y \, dx - \sin x \sin y \, dy$$

where $\mathbf{c}(t) = (t, t^2)$, $0 \leq t \leq 1$. (Use Corollary 1, by finding a function $f(x, y)$ whose gradient is $\nabla f = (\cos x \cos y, -\sin x \sin y)$. Then just evaluate f at the endpoints.)

5. Integrals of functions on paths

Sometimes you just want to integrate a function $f(x, y)$ along a path $\mathbf{c}(t) = (x(t), y(t))$. Integrating the constant function $f = 1$ should give you the length of the path. We know the length of the path for $a \leq t \leq b$ is

$$\text{Length} = \int_a^b \sqrt{(x'(u))^2 + (y'(u))^2} \, du.$$

Let $s(t)$ be the length on the path that we have travelled up to time t , starting at $t = a$. Then

$$s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2} \, du,$$

so

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2},$$

so

$$(5a) \quad ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \|\mathbf{c}'(t)\| \, dt.$$

This formula for ds is all you need to integrate a function on a path. The integral of $f(x, y)$ on \mathbf{c} , for $a \leq t \leq b$, is defined to be

$$\int_{\mathbf{c}} f \, ds = \int_a^b f(\mathbf{c}(t)) \sqrt{(x'(t))^2 + (y'(t))^2} \, dt.$$

In other words, just plug \mathbf{c} into f , and remember the formula (5a) for ds .

The integral $\int_{\mathbf{c}} f \, ds$ gives the total amount of f on \mathbf{c} . For example, if the path is a wire, and f is a charge distribution or a density, then $\int_{\mathbf{c}} f \, ds$ is the total charge on the wire, or the total mass of the wire.

The **average** of f on \mathbf{c} is

$$\text{Average of } f = \frac{1}{L} \int_{\mathbf{c}} f \, ds,$$

where L is the length of \mathbf{c} .

Exercise 5.1 Let $\mathbf{c}(t) = (a + r \cos t, b + r \sin t)$, $0 \leq t \leq 2\pi$. Compute the integrals

a) $\int_{\mathbf{c}} xy \, ds,$

b) $\int_{\mathbf{c}} x^2 - y^2 \, ds$

c) $\int_{\mathbf{c}} x^3 - 3xy^2 \, ds.$

Notice anything? Does the same thing happen for other functions?

Exercise 5.2 We know from one-variable calculus that

$$\int_0^{2\pi} \cos^{2n} x \, dx = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} 2\pi.$$

Use this formula to compute the average of the function $f(x, y) = x^{2n}$ over the unit circle centered at $(0, 0)$.

6. Flux Integrals

Return now to vector fields. Instead of measuring how much \mathbf{F} flows *along* \mathbf{c} , we can measure how much \mathbf{F} flows *through* \mathbf{c} . The basic idea is simple: At each point p on the path, we have a vector $\mathbf{F}(p)$, and a normal vector $\mathbf{N}(p)$ to the path at p . The dot product $\mathbf{F}(p) \cdot \mathbf{N}(p)$ measures how much $\mathbf{F}(p)$ is pointing in the direction of $\mathbf{N}(p)$. To find the total amount of flow of \mathbf{F} through \mathbf{c} , we compute the integral

$$\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{N} \, ds,$$

using the formulas in the previous section. This is called a **Flux Integral**. You compute it explicitly as follows.

First, you have to find \mathbf{N} . Recall the tangent vector to \mathbf{c} is $\mathbf{c}' = (x', y')$. Therefore the vector $(-y', x')$ is perpendicular to \mathbf{c}' , hence is normal to the path. (When you cross \mathbf{c}' into this normal vector, your right thumb points up.) To make it a unit vector we divide by the length (which is the same length as \mathbf{c}'), and define

$$\mathbf{N}(t) = \frac{1}{\|\mathbf{c}'(t)\|} (-y', x').$$

At each time t , $\mathbf{N}(t)$ is a unit vector which is normal to the path at the point $\mathbf{c}(t)$. At the point $\mathbf{c}(t)$, we have

$$\mathbf{F} \cdot \mathbf{N} = \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{N}(t) = \mathbf{F}(\mathbf{c}(t)) \cdot \frac{1}{\|\mathbf{c}'(t)\|} (-y', x').$$

so the explicit formula for the flux integral is

$$\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{N} \, ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \frac{1}{\|\mathbf{c}'(t)\|} (-y'(t), x'(t)) \|\mathbf{c}'(t)\| \, dt = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot (-y'(t), x'(t)) \, dt. \blacksquare$$

In other words, if $\mathbf{F} = (P, Q)$, then

$$\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{\mathbf{c}} Q \, dx - P \, dy.$$

Example: Take $\mathbf{F} = (x, y)$, $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$. Then the flux of \mathbf{F} through \mathbf{c} is

$$\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{N} \, ds = \int_0^{2\pi} (\cos t, \sin t) \cdot (-\cos t, \sin t) \, dt = -2\pi.$$

The answer is negative because our \mathbf{N} points inward, while \mathbf{F} is exploding outward, so there is negative flux.

Exercise 6.1 Calculate the flux of $\mathbf{F} = (x, y)$ through the circle of radius 1 centered at the point $(1, 2)$.