Solutions to exercises in Note 2

Line integrals in the plane, vector fields, work integrals, conservative vector fields and independence of path, functions on curves, flux integrals

1. Line Integrals in the Plane

Exercise 1.1 Compute $\int_{\mathbf{c}} x\ dy$ over the following counterclockwise curves, and then compute the area inside of each curve.

a) $\mathbf{c}$ is the triangle from $(0,0)$ to $(a,0)$ to $(0,b)$ back to $(0,0)$.

Solution :

$$\int_{\mathbf{c}} x\ dy = \int_{\mathbf{c}_1} x\ dy + \int_{\mathbf{c}_2} x\ dy + \int_{\mathbf{c}_3} x\ dy,$$

where

$$\mathbf{c}_1(t) = (t,0),\ 0 \le t \le a, \quad \mathbf{c}_2(t) = (a - ta, tb),\ 0 \le t \le 1, \quad \mathbf{c}_3(t) = (0, b - tb),\ 0 \le t \le 1.$$

For $\mathbf{c}_1$ we have $dy = 0$. For $\mathbf{c}_3$ we have $x = 0$. So

$$\int_{\mathbf{c}} x\ dy = \int_{\mathbf{c}_2} x\ dy = \int_0^1 (a - ta)b\ dt = ab/2.$$

This is the area inside $\mathbf{c}$.

b) $\mathbf{c}$ follows the graph of $y = x^2$ from $(0,0)$ to $(1,1)$, then follows the graph of $x = y^2$ from $(1,1)$ back to $(0,0)$.

Solution :

$$\mathbf{c}_1(t) = (t,t^2),\ 0 \le t \le 1, \quad \mathbf{c}_2(t) = (t^2,t),\ 0 \le t \le 1,$$

so

$$\int_{\mathbf{c}} x\ dy = \int_{\mathbf{c}_1} x\ dy - \int_{\mathbf{c}_2} x\ dy = \int_0^1 t \cdot 2t\ dt - \int_0^1 t^2\ dt = 1/3.$$

The area inside $\mathbf{c}$ is given by the integral

$$\int_0^1 \sqrt{x} - x^2\ dx = 2/3 - 1/3 = 1/3.$$

Exercise 1.2 Compute $\int_{\mathbf{c}} x\ dx + y\ dy$ along the curves

$$\mathbf{c}_1(t) = (t,t^2), \quad \mathbf{c}_2(t) = (t,t), \quad \mathbf{c}_3(t) = (t^2,t),$$

all with $0 \le t \le 1$. You should get the same answer each time.

Solution :

$$\int_{\mathbf{c}_1} x\ dx + y\ dy = \int_0^1 t + t^2 \cdot 2t\ dt = 1.$$

The other integrals are similar, and equal 1.

Exercise 1.3 Compute $\int_{\mathbf{c}} -y\ dx + x\ dy$ along the three curves in the previous problem. You will get different answers this time.

Solution :

$$\int_{\mathbf{c}_1} -y\ dx + x\ dy = \int_0^1 -t^2 + t \cdot 2t\ dt = 1/3.$$

$$\int_{\mathbf{c}_2} -y\ dx + x\ dy = \int_0^1 -t + t\ dt = 0.$$

$$\int_{\mathbf{c}_3} -y\ dx + x\ dy = \int_0^1 -t \cdot 2t + t^2\ dt = -1/3.$$

2. Vector Fields

no exercises in this section

3. Work Integrals

Exercise 3.1 Compute $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c}$ for

a) $\mathbf{F} = (x, y)$, $\quad$ $\mathbf{c}$ is the line segment from $(1,0)$ to $(0,2)$

Solution :

$$\mathbf{c}(t) = (1,0) + t(-1,2) = (1-t, 2t).$$

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_0^1 (1-t, 2t) \cdot (-1,2)\ dt = \int_0^1 5t - 1\ dt = 3/2.$$

b) $\mathbf{F} = (x, y)$, $\quad$ $\mathbf{c}(t) = (\cos t, 2\sin t),\ 0 \le t \le \frac{\pi}{2}$.

Solution :

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_0^{\pi/2} (\cos t, 2\sin t) \cdot (-\sin t, 2\cos t)\ dt = \int_0^{\pi/2} 3\sin t \cos t\ dt = 3/2.$$

c) $\mathbf{F} = (-y, x)$, $\mathbf{c}$ as in a).

Solution :

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_0^1 (-2t, 1-t) \cdot (-1,2)\ dt = 2.$$

d) $\mathbf{F} = (-y, x)$, $\mathbf{c}$ as in b).

Solution :

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_0^{\pi/2} (-2\sin t, \cos t) \cdot (-\sin t, 2\cos t)\ dt = \pi.$$

(You should get the same answers for a) and b), but different answers for c) and d). This is connected to the fact that $\mathbf{F} = (x, y)$ is conservative, while $\mathbf{F} = (-y, x)$ is not conservative. See also exercises 1.2,3 above.)

Exercise 3.2 Compute $\int_{\mathbf{c}} F \cdot d\mathbf{c}$ for $F = (3x - 4y, 4x + 2y)$, where $\mathbf{c}$ is the counterclockwise ellipse with equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Solution :

$$\mathbf{c}(t) = (2\cos t, 3\sin t), \quad 0 \le t \le 2\pi.$$

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{c} = \int_0^{2\pi} (3x - 4y, 4x + 2y) \cdot (-2\sin t, 3\cos t) dt = 48\pi.$$

Exercise 3.3 The World's Most Interesting Vector Field is

$$F(x,y) = (\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}).$$

Compute $\int_{\mathbf{c}} F \cdot d\mathbf{c}$ over the following paths $\mathbf{c}$:

a) $\mathbf{c}(t) = (\cos t, \sin t), \quad 0 \le t \le \frac{\pi}{2}$

b) The line segment from $(1,0)$ to $(1,1)$ followed by the line segment from $(1,1)$ to $(0,1)$

c) $\mathbf{c}(t) = (\cos t, \sin t), \quad 0 \le t \le \pi$ (upper semicircle)

d) $\mathbf{c}(t) = (\cos t, -\sin t), \quad 0 \le t \le \pi$ (lower semicircle).

You should get the same answers for a) and b), but different answers for c) and d). What do you guess is the cause of this different behavior?

Solution : a) and b) are $\pi/2$, c) is π, and d) is $-\pi$. It has to do with whether or not the two paths together make a loop around $(0,0)$.

4. Conservative Vector Fields and Independence of Path

Exercise 4.1 Compute

$$\int_{\mathbf{c}} \cos x \cos y \; dx - \sin x \sin y \; dy$$

where $\mathbf{c}(t) = (t, t^2)$, $0 \le t \le 1$. (Use Corollary 1, by finding a function $f(x, y)$ whose gradient is $\nabla f = (\cos x \cos y, -\sin x \sin y)$. Then just evaluate f at the endpoints.)
Solution : $f(x, y) = \sin x \cos y$. $\mathbf{c}(0) = (0, 0)$ and $\mathbf{c}(1) = (1, 1)$, so

$$\int_{\mathbf{c}} F \cdot d\mathbf{c} = f(1, 1) - f(0, 0) = \sin 1 \cos 1.$$

5. Integrals of functions on paths

Exercise 5.1 Let $\mathbf{c}(t) = (a + r \cos t, b + r \sin t)$, $0 \le t \le 2\pi$. Compute the integrals
a) $\int_{\mathbf{c}} xy \; ds$,
b) $\int_{\mathbf{c}} x^2 - y^2 \; ds$
c) $\int_{\mathbf{c}} x^3 - 3xy^2 \; ds$.
Notice anything? Does the same thing happen for other functions?
Solution : In each case, the integral is $\pi r^2 f(a, b)$. This only happens with *harmonic* functions, ie, those for which $f_{xx} + f_{yy} = 0$.

Exercise 5.2 We know from one-variable calculus that

$$\int_0^{2\pi} \cos^{2n} x \; dx = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} 2\pi.$$

Use this formula to compute the average of the function $f(x, y) = x^{2n}$ over the unit circle centered at $(0, 0)$.
Solution :

$$\frac{1}{2\pi} \int_C x^{2n} \; ds = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}.$$

6. Flux Integrals

Exercise 6.1 Calculate the flux of $\mathbf{F} = (x, y)$ through the circle of radius 1 centered at the point (1,2).
Solution :

$$\mathbf{c}(t) = (1 + \cos t, 2 + \sin t), \quad 0 \le t \le 2\pi.$$

$$\int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{N} \; ds = \int_0^{2\pi} (1 + \cos t, 2 + \sin t) \cdot (-\cos t, -\sin t) \; dt = -2\pi.$$

Notice that the center of the circle does not matter. The exploding vector field has the same flux through any unit circle.