**Note 3**

Double integrals, Area, Average, Center of Mass

## 1. What is a Double Integral?

Take a region $R$ in the $xy$ plane. For example, $R$ could be a solid rectangle or a solid disk. Now take a function $f(x, y)$. First suppose $f$ is positive everywhere on $R$. The graph of $f$ is then a surface sitting above $R$. The **double integral** of $f$ over $R$ is

$$\iint_R f dR = \text{Volume of the space above } R\text{, under the graph of } f.$$

If $f$ is not positive everywhere on $R$, then $\iint_R dR$ is the volume between the graph of $f$ and the region $R$, where the part below $R$ counts as negative volume.

Thus, for example, if $R$ is a solid disk of any radius centered at $(0, 0)$, and $f(x, y) = xy$ (a saddle), then there is as much volume below $R$ as above $R$, so $\iint_R xy\ dR = 0$, if $R$ is this disk. If $R$ is just a quarter disk in the quadrant $x > 0, y > 0$, then $\iint_R xy\ dR > 0$, and we will see how to compute it exactly in a moment.

For another example, suppose $R$ is any region, and $f(x, y) = 1$ (a constant function). Then the space under the graph of $f$ is a piece of a jig-saw puzzle one unit thick, in the shape of $R$. The volume of this space is the area of $R$. That is,

$$\iint_R dR = \text{Area of } R.$$

If $f(x, y) = 5$, then

$$\iint_R 5\ dR = 5 \times \text{Area of } R,$$

and similarly for any constant function $f$.

## 2. How to compute a double integral over a rectangle

Suppose $R$ is a rectangle defined by the inequalities $a \le x \le b,\ c \le y \le d$. The sides of $R$ must be parallel to the $x$ and $y$ axes. Then there are two ways to compute $\iint_R f\ dR$. The first way:

$$\iint_R f\ dR = \int_a^b \int_c^d f(x, y)\ dydx.$$

Do the inner integral $\int_c^d f(x, y)\ dy$ by holding $x$ constant, and integrating with respect to $y$. After you integrate, and plug in the limits $c, d$ for $y$, you will only have $x's$ left. Then do the outer integral, from $a$ to $b$ of this function of $x$.

Example: Let $R$ be the rectangle $2 \le x \le 3,\ 0 \le y \le 1$, and take $f = x + y^2$. Then

$$\iint_R x + y^2\ dR = \int_2^3 \int_0^1 x + y^2\ dydx = \int_2^3 [xy + \frac{y^3}{3}]_{y=0}^{y=1}\ dx = \int_2^3 x + \frac{1}{3}\ dx = \frac{17}{6}.$$

The second way is

$$\iint_R f\ dR = \int_c^d \int_a^b f(x,y)\ dxdy.$$

Always do the inside integral first. This time $x$ will disappear when you plug in the limits $a, b$. You are left with a function of $y$, which you then integrate from $c$ to $d$.

$$\int_0^1 \int_2^3 x + y^2\ dxdy = \int_0^1 [\frac{x^2}{2} + xy^2]_{x=2}^{x=3}\ dy = \int_0^1 \frac{5}{2} + y^2\ dy = \frac{17}{6}.$$

It doesn't matter which way you do it, you'll get the same answer both ways.

**Exercise 1.1** Compute the double integrals $\iint_R f\ dR$, where $R$ is the square $0 \leq x \leq 1,\ 0 \leq y \leq 1$, for the following functions $f(x,y)$

a) $f(x,y) = xy$
b) $f(x,y) = x^3 - 3xy^2$.
c) $f(x,y) = \sin(\pi x)\sin(\pi y)$
d) $f(x,y) = \sin(\pi x) + \cos(\pi y)$
e) $f(x,y) = e^{x+y}$
f) $f(x,y) = ye^{x+y}$.

To check your answers for a), c), e), f), do the following. In these four integrals, we have $f(x,y) = g(x)h(y)$, for some other functions $g(x), h(y)$. Compute, in each case, the product of single integrals $(\int_0^1 g(x)dx)(\int_0^1 h(y)dy)$. You should get the same answers as the double integrals you computed in parts a),c),e),f). This does not quite work for parts b) and d) because the function $f(x,y)$ is not a product of a function of $x$ and a function of $y$. But b) and d) are sums of product functions, and you can apply this check on each term. To be clear, here is the formula we are using.

**Product formula for a Rectangle.** *If $f(x,y) = g(x)h(y)$, and $R$ is the rectangle $a \leq x \leq b,\ c \leq y \leq c$, then*

$$\iint_R f(x,y)\ dR = (\int_a^b g(x)\ dx)(\int_c^d h(y)\ dy).$$

To use this formula, the function *must* be a product of a function of $x$ times a function of $y$, and the region $R$ *must* be a rectangle with sides parallel to the $x$ and $y$ axes.

**Exercise 1.2** Prove the above product formula.

## 3. How to compute a double integral over a disk

Suppose $R$ is a disk of radius $a$, centered at the origin $(0,0)$. Usually the best way to integrate over $R$ is to use polar coordinates, as follows. You have a function $f(x,y)$. First, set

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad dR = rdrd\theta.$$

Then

$$\iint_R f\ dR = \int_0^{2\pi} \int_0^a f(r\cos\theta, r\sin\theta)\ rdrd\theta.$$

The reason for putting $r$ in front of $dr$ will be explained later. Note that the integral of $f(x, y)$ over a disk in the $xy$-plane equals the integral of the function $rf(r\cos\theta, r\sin\theta)$ over a rectangle in the $r, \theta$-plane.

Example 1: Let $a = 3$, and take $f = x + y^2$. Then

$$\begin{aligned}\iint_R x + y^2\ dR &= \int_0^{2\pi} \int_0^3 (r\cos\theta + r^2\sin^2\theta)\ rdrd\theta \\ &= \int_0^{2\pi} (9\cos\theta + \frac{81}{4}\sin^2\theta)\ d\theta \\ &= \frac{81\pi}{4}.\end{aligned}$$

Note that the integral involving $\cos\theta$ was zero. That's because the graph of $x$ is a plane, and there is equal space between the graph of $x$ above and below the $xy$-plane, above a disk centered at (0,0). Another way to see this is using the center of mass, discussed in the next section.

**Exercise 2.1** Let $R$ be the disk of radius 1 centered at $(0, 0)$. Compute $\iint_R f\ dR$ for the following functions $f(x, y)$.

a) $f(x, y) = x^2$.
b) $f(x, y) = xy$
c) $f(x, y) = 1 - x^2 - y^2$.

**Exercise 2.2** Compute

$$\iint_R e^{x^2+y^2}\ dR$$

where $R$ is the disk of radius $a$ centered at $(0, 0)$.

The same kind of formula works for certain pieces of a disk, as long as the corresponding region in the $r\theta$ plane is still a rectangle.

Example 2: Take an annulus (a washer) with inner radius 1 and outer radius 2, and let $R$ be the quarter of this annulus which lies in the first quadrant. So $R$ is described by the inequalities $1 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$. This is a rectangle in the $r\theta$ plane. Let $f(x, y)$ be any function. Then

$$\iint_R f\ dR = \int_0^{\frac{\pi}{2}} \int_1^2 f(r\cos\theta, r\sin\theta)\ rdrd\theta.$$

**Exercise 2.3** Take $f = 1$, and compute $\iint_R dR$ where $R$ is the region in Example 2. You can check your answer by computing the area of $R$ using basic geometry.

## 4. Average and Center of Mass

In this section, $R$ can be any region in the $xy$-plane. Recall that the area of $R$ is

$$\text{Area}(R) = \iint_R dR.$$

Now if $f(x, y)$ is a function, then the **average** of $f$ on $R$ is

$$\text{Average of } f \text{ on } R = \frac{1}{\text{Area}(R)} \iint_R f\ dR.$$

This is another way to interpret $\iint_R f\, dR$, in addition to the volume interpretation.

**Exercise 4.1** Find the average of the function $x^2$ over
a) The unit disk centered at (0,0)
b) The square with corners at $(\pm 1, \pm 1)$.

Now, the **center of mass** of $R$ is the point $(\bar{x}, \bar{y})$ obtained as follows. The $x$-coordinate $\bar{x}$ of the center of mass is the average of all of the $x$-coordinates of points in $R$. That is, $\bar{x}$ is the average of the function $f = x$ on $R$. Same for $y$. Thus we have the formulas

$$\bar{x} = \frac{1}{\text{Area}(R)} \iint_R x\, dR, \qquad \bar{y} = \frac{1}{\text{Area}(R)} \iint_R y\, dR.$$

**Exercise 4.2** Find the center of mass of the quarter annulus $1 \le r \le 2$, $0 \le \theta \le \frac{\pi}{2}$ discussed in Sec. 3, Example 2.

Sometimes you can turn this around and use the center of mass to compute certain integrals. That is, maybe the $x$-coordinate of the center of mass is obvious. Then you can integrate $x$ and $y$ over $R$ by means of the formula

$$\iint_R x\, dR = \bar{x} \cdot \text{Area}(R).$$

Example 1: $R$ is the disk of radius 2 centered at $(3, 7)$. Compute $\iint_R x\, dR$. Answer: The center of mass of a disk is the center of the disk, so $\bar{x} = 3$. The area of $R$ is $4\pi$, so

$$\iint_R x\, dR = 4\pi \cdot 3 = 12\pi, \quad \text{and} \iint_R y\, dR = 4\pi \cdot 7 = 28\pi$$

Example 2: Compute the integral of $f(x, y) = 7 - 3x + 2y$ over the inside of the ellipse

$$\frac{(x-1)^2}{4} + \frac{(y+5)^2}{9}.$$

Answer: The center of mass of $R$ is obviously $(1, -5)$. The area is $2 \cdot 3 \cdot \pi = 6\pi$. Therefore

$$\iint_R 7 - 3x + 2y\, dR = 7 \cdot 6\pi - 3 \cdot 1 \cdot 6\pi + 2 \cdot (-5) \cdot 6\pi = -36\pi.$$

The point is that you can sometimes integrate the functions $1, x, y$ just by looking at $R$. And then you can integrate something like $7 - 3x + 2y$ by integrating each term separately. This only works for regions where you can spot the center of mass without computation.

**Exercise 4.3** Use the method of examples 1,2 to compute the integral $\iint_R 7 - 3x + 2y\, dR$ for the following regions $R$
a) $R$ is the disk of radius 5 centered at $(1, 2)$.
b) $R$ is a square of side length 5 centered at $(1, 2)$. (It doesn't matter how the square is rotated!).

You can check your answers by integrating the function $7 - 3(x+1) + 2(y+1)$ over the same regions shifted to be centered at (0,0), using the methods of sections 1,2 for integrating any function over a disk or a square.