Note 3

Double integrals, Area, Average, Center of Mass

1. What is a Double Integral?

No exercises in this section.

2. How to compute a double integral over a rectangle

Exercise 1.1 Compute the double integrals $\iint_R f\ dR$, where R is the square $0 \le x \le 1,\ 0 \le y \le 1$, for the following functions $f(x,y)$

a) $f(x,y) = xy$
b) $f(x,y) = x^3 - 3xy^2$.
c) $f(x,y) = \sin(\pi x)\sin(\pi y)$
d) $f(x,y) = \sin(\pi x) + \cos(\pi y)$
e) $f(x,y) = e^{x+y}$
f) $f(x,y) = ye^{x+y}$.

Answers: a) 1/4, b) -1/4, c) $4/\pi$, d) $2/\pi$, e) $(e-1)^2$, f) $e-1$.

Product formula for a Rectangle. *If $f(x,y) = g(x)h(y)$, and R is the rectangle $a \le x \le b,\ c \le y \le c$, then*

$$\iint_R f(x,y)\ dR = (\int_a^b g(x)\ dx)(\int_c^d h(y)\ dy).$$

To use this formula, the function *must* be a product of a function of x times a function of y, and the region R *must* be a rectangle with sides parallel to the x and y axes.

Exercise 1.2 Prove the above product formula.

Solution:

$$\iint_R f(x,y)\ dR = \int_a^b \int_c^d g(x)h(y)\ dydx = \int_a^b g(x)\big(\int_c^d h(y)\ dy\big)dx = \big(\int_c^d h(y)\ dy\big)\big(\int_a^b g(x)\ dx\big).$$

3. How to compute a double integral over a disk

Exercise 2.1 Let R be the disk of radius 1 centered at $(0,0)$. Compute $\iint_R f\ dR$ for the following functions $f(x,y)$.

a) $f(x,y) = x^2$.
b) $f(x,y) = xy$
c) $f(x,y) = 1 - x^2 - y^2$.

Answers: a) $\pi/4$, b) 0, c) $\pi/2$

Exercise 2.2 Compute

$$\iint_R e^{x^2+y^2}\ dR$$

where R is the disk of radius a centered at $(0,0)$.

Answer: $\pi(e^{a^2} - 1)$.

Exercise 2.3 Take $f = 1$, and compute $\iint_R dR$ where R is the region in Example 2. You can check your answer by computing the area of R using basic geometry.

Answer: $3\pi/4$.

4. Average and Center of Mass

Exercise 4.1 Find the average of the function x^2 over

a) The unit disk centered at (0,0)

b) The square with corners at $(\pm 1, \pm 1)$.

Answers: a) 1/4, b) 1/3

Exercise 4.2 Find the center of mass of the quarter annulus $1 \leq r \leq 2$, $0 \leq \theta \leq \frac{\pi}{2}$ discussed in Sec. 3, Example 2.

Answer: $(\frac{2}{\pi}, \frac{2}{\pi})$.

Exercise 4.3 Use the method of examples 1,2 to compute the integral $\iint_R 7 - 3x + 2y \; dR$ for the following regions R

a) R is the disk of radius 5 centered at $(1, 2)$.

b) R is a square of side length 5 centered at $(1, 2)$. (It doesn't matter how the square is rotated!).

Answers: a) 200π, b) 200.